Physics 8/9, fall 2019, equation sheet

work in progress

(Chapter 1: foundations)

1 year $\approx 3.156 \times 10^7$ s

circumference of Earth $\approx 40 \times 10^6$ m   (radius = 6378 km)

mass of Earth $\approx 6.0 \times 10^{24}$ kg

speed of light $c = 2.9979 \times 10^8$ m/s

mass of proton or neutron $\approx$ 1 amu ("atomic mass unit") $= 1\ \frac{\text{g}}{\text{mol}} = \frac{0.001\ \text{kg}}{6.022\times 10^{23}} = 1.66 \times 10^{-27}$ kg

Some exact definitions: 1 inch = 0.0254 meter. 1 foot = 12 inches. 1 mile = 5280 feet.

Weight of 1 kg = 2.205 pounds.

Other unit conversions: try typing e.g. "1 mile in centimeters" or "1 gallon in liters" into google!

Unit conversions: The trick is to make use of the fact that a ratio of two equal values, like $\frac{1\,\text{inch}}{2.54\,\text{cm}}$, equals 1. So you "multiply by 1" and then cancel the unwanted units until you are left with the desired units. For example, to convert 1 mile into meters, we can write [notice that each ratio in parentheses equals 1]

$$1\,\text{mile} \times \left(\frac{5280\,\text{foot}}{\text{mile}}\right) \times \left(\frac{12\,\text{inch}}{\text{foot}}\right) \times \left(\frac{0.0254\,\text{m}}{\text{inch}}\right) = 1609.3\,\text{m} \ .$$

Significant digits: If I write that the mass of my golden retriever, Alfie, is $m = 37.73\,\text{kg}$, the implication, according to convention, is that I know that $37.725\,\text{kg} \le m \le 37.735\,\text{kg}$. Since a meal or a visit to the back yard can easily change Alfie's mass by about 0.1 kg or so, it seems more realistic for me to quote Alfie's mass as $m = 37.7\,\text{kg}$, which implies that I know his mass to roughly $\pm 0.1\%$. If it has been a week or two since I last picked Alfie up and stood on the scale with him, it may be more honest for me to write $m = 38\,\text{kg}$. So depending on the circumstances, $m = 38\,\text{kg}$ or $m = 37.7\,\text{kg}$ may best convey my knowledge of Alfie's mass. The key idea is that, by convention, the number of digits you write down for a given quantity makes an implicit statement about how well you know that quantity. If all of the inputs to a given calculation are known to 3 significant digits, then it is reasonable for your result to be stated to 3 or 4 significant digits, but not to 5 or 6 significant digits. One word of caution: when working through a calculation, you usually want to keep a few extra digits in your intermediate results, then round your final result to the stated precision at the end. Also, a value like 100 kg, with trailing zeros but no decimal point, has ambiguous precision; in this unusual case, if you need to be unambiguous, use scientific notation: 1×10^2 kg or 1.0×10^2 kg or 1.00×10^2 kg; another trick is to write 100. kg to imply that all of the trailing zeros are significant, which in this case is the same as 1.00×10^2 kg. When writing homework problems, I usually avoid numbers like 100 kg, and I go out of my way instead to write 101 kg or 100.0 kg. Usually if I write "38000 feet" I mean 2 significant digits, whereas if I write "38001 feet" I mean 5 significant digits; Mazur's textbook, by contrast, means 5 significant digits in both cases. (I try my best to avoid contradicting the textbook, and those few times when I do, I aim to tell you that I am doing so.)

A special case: In this course, we do not treat the gravitational acceleration $g = 9.8\,\mathrm{m/s^2}$ as an input having only two significant figures. Though at different points on Earth, the acceleration due to gravity varies between 9.764 and $9.834\,\mathrm{m/s^2}$, we will use g in calculations as if it were specified as $9.80\,\mathrm{m/s^2}$, so that g does not limit our ability to solve problems using 3 significant figures.

(Chapter 2: motion in one dimension)

x component of displacement: $\Delta x = x_f - x_i$ where "f" is for (f)inal, and "i" is for (i)nitial.

If an object goes from x_i to x_f, changing direction at intermediate points x_a and x_b, then distance traveled (in one dimension) is $d = |x_a - x_i| + |x_b - x_a| + |x_f - x_b|$

x component of (instantaneous) velocity: $v_x = \frac{\mathrm{d}x}{\mathrm{d}t}$

Speed (a scalar) is the magnitude of velocity (a vector). In one dimension, $v = |v_x|$

average velocity $= \frac{\text{displacement}}{\text{time interval}}$ $\qquad v_{x,\mathrm{av}} = \frac{x_f - x_i}{t_f - t_i}$

average speed $= \frac{\text{distance traveled}}{\text{time interval}}$

Solving quadratic equations: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(Chapter 3: acceleration)

Because we will use exclusively one axis (called the x-axis) for the first nine chapters of Mazur's textbook, we need to introduce two conventions that may seem confusing to you if you took high-school physics. First: for free-fall problems in one dimension (e.g. you drop a ball, or you toss a ball directly upward in a motion that is perfectly vertical), the x-axis will point *upward.* Second: for inclined-plane problems (e.g. a briefcase slides down an icy driveway), the x-axis will point *downhill.*

When discussing gravity near Earth's surface, we introduce a constant $g = 9.8\ \mathrm{m/s^2}$, which is "the acceleration due to Earth's gravity." If the x axis points *upward*, then $a_x = -g$ for free fall near Earth's surface. So for this scenario we use a constant negative value for a_x.

If the x axis points *downhill* along an inclined plane, then $a_x = g \sin\theta$ for an object sliding down the inclined plane (inclined at angle θ w.r.t. horizontal). So for this scenario we use a constant positive value for a_x. Galileo studied motion on an inclined plane so that the magnitude of a_x would be small enough to allow detailed measurements to be made by eye.

For constant acceleration:

$$v_{x,f} = v_{x,i} + a_x t$$

$$x_f = x_i + v_{x,i} t + \frac{1}{2} a_x t^2$$

$$v_{x,f}^2 = v_{x,i}^2 + 2 a_x \ (x_f - x_i)$$

The third equation comes from combining the first two equations and eliminating t.

(Chapter 4: momentum)

Momentum is $\vec{p} = m\vec{v}$. Momentum is constant for an *isolated* system. A system is isolated if there are no external pushes or pulls (later we'll say "forces") applied to the system. Conservation of momentum in isolated two-body collision implies

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 v_{2x,f}$$

which then implies (for isolated system, two-body collision)

$$\frac{\Delta v_{1x}}{\Delta v_{2x}} = -\frac{m_2}{m_1}$$

If system is not isolated, then we *cannot* write $\vec{p}_f - \vec{p}_i = 0$. Instead, we give the momentum imbalance caused by the external influence a name ("impulse") and a label ($\vec{J}$). Then we can write $\vec{p}_f - \vec{p}_i = \vec{J}$.

(Chapter 5: energy)

In chemistry, a calorie is 1 cal = 4.18 J. In nutrition, a "food Calorie" is 1 Cal = 4180 J.

The energy of motion is *kinetic* energy:

$$K = \frac{1}{2}mv^2$$

For an *elastic* collision, kinetic energy K is constant. For a two-body elastic collision, the *relative speed* is unchanged by the collision, though obviously the relative velocity changes sign. Thus, for a for a two-body elastic collision along the x axis (Eqn. 5.4),

$$(v_{1x,f} - v_{2x,f}) = -(v_{1x,i} - v_{2x,i})$$

.

For a *totally inelastic collision*, the two objects stick together after collision: $\vec{v}_{1f} = \vec{v}_{2f}$. This case is easy to solve, since one variable is eliminated.

In the real world (but not in physics classes), most collisions are *inelastic* but are not totally inelastic. K is not constant, but $v_{12,f} \neq 0$. So you can define a *coefficient of restitution*, e, such that $e = 1$ for elastic collisions, $e = 0$ for totally inelastic collisions, and $0 < e < 1$ for inelastic collisions. Then you can write (though it is seldom useful to do so)

$$(v_{1x,f} - v_{2x,f}) = -e\ (v_{1x,i} - v_{2x,i})$$

If you write down the momentum-conservation equation (assuming that system is isolated, so that momentum is constant) for a two-body collision along the x axis,

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 v_{2x,f}$$

and the equation that kinetic energy is also constant in an elastic collision,

$$\frac{1}{2}m_1 v_{1x,i}^2 + \frac{1}{2}m_2 v_{2x,i}^2 = \frac{1}{2}m_1 v_{1x,f}^2 + \frac{1}{2}m_2 v_{2x,f}^2$$

you can (with some effort) solve these two equations in two unknowns. The quadratic equation for energy conservation gives *two* solutions, which are equivalent to

$$(v_{1x,f} - v_{2x,f}) = \pm(v_{1x,i} - v_{2x,i})$$

In the "+" case, the two objects miss each other, as if they were two trains passing on parallel tracks. The "−" case is the desired solution. In physics, the "other" solution usually means *something*, even if it is not the solution you were looking for.

(Chapter 6: relative motion)

Center of mass:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

Center of mass velocity (equals velocity of ZM frame):

$$v_{ZM,x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

(The following chapter 6 results are less important, but I list them here anyway.)

Convertible kinetic energy: $K_{\text{conv}} = K - \frac{1}{2} m v_{CM}^2$

Elastic collision analyzed in ZM ("$*$") frame:

$$v_{1i,x}^* = v_{1i,x} - v_{ZM,x}, \quad v_{2i,x}^* = v_{2i,x} - v_{ZM,x}$$
$$v_{1f,x}^* = -v_{1i,x}^*, \quad v_{2f,x}^* = -v_{2i,x}^*$$
$$v_{1f,x} = v_{1f,x}^* + v_{ZM,x}, \quad v_{2f,x} = v_{2f,x}^* + v_{ZM,x}$$

Inelastic collision analyzed in ZM frame (restitution coefficient e):

$$v_{1f,x}^* = -e v_{1i,x}^*, \quad v_{2f,x}^* = -e v_{2i,x}^*$$

(Chapter 7: interactions)

For two objects that form an isolated system (i.e. interacting only with one another), the ratio of accelerations is

$$\frac{a_{1x}}{a_{2x}} = -\frac{m_2}{m_1}$$

When an object near Earth's surface moves a distance Δx in the direction away from Earth's center (i.e. upward), the change in gravitational potential energy of the Earth+object system is

$$\Delta U = mg\Delta x$$

(Chapter 8: force)

The rate of change of the momentum of object A is the vector sum of forces exerted **on** object A.

Force (newtons: $1\ \mathrm{N} = 1\ \mathrm{kg}\cdot\mathrm{m/s^2}$) is rate of change of momentum:

$$\sum \vec{F}_{\mathrm{on\,A}} = \frac{\mathrm{d}\vec{p_A}}{\mathrm{d}t}$$

Impulse (change in momentum due to external force):

$$\vec{J} = \Delta\vec{p} = \int \vec{F}_{\mathrm{ext}}\ \mathrm{d}t$$

Equation of motion for a single object A:

$$\sum \vec{F}_{\mathrm{on\,A}} = m_A\ \vec{a}_A$$

Equation of motion for CoM of several objects (depends only on forces exerted by objects external to the system on objects inside the system, i.e. the vector sum of external forces):

$$\sum \vec{F}_{\mathrm{ext}} = m_{\mathrm{total}}\ \vec{a}_{CM}$$

Gravitational potential energy near earth's surface (h = height):

$$U_{\mathrm{gravity}} = mgh$$

Force of gravity near earth's surface (force is $-\frac{\mathrm{d}U_{\mathrm{gravity}}}{\mathrm{d}x}$):

$$F_x = -mg$$

Potential energy of a spring:

$$U_{\text{spring}} = \frac{1}{2}k(x - x_0)^2$$

where x_0 is the "relaxed length" of the spring, and k is the "spring constant" (units for k are newtons per meter).

Hooke's Law (force is $-\frac{\mathrm{d}U_{\text{spring}}}{\mathrm{d}x}$):

$$F_{\text{by spring ON load}} = -k(x - x_0)$$

(Chapter 9: work)

Work done on a system by an external, nondissipative force in one dimension:

$$W = \int_{x_i}^{x_f} F_x(x)dx$$

which for a constant force reduces to

$$W = F_x \Delta x$$

Power is rate of change of energy (measured in watts: 1 W = 1 J/s):

$$P = \frac{\mathrm{d}E}{\mathrm{d}t}$$

In one dimension, power delivered by constant external force is

$$P = F_{\text{ext},x} v_x$$

(Chapter 10: motion in a plane)

Various ways to write a vector:

$$\vec{A} = (A_x, A_y) = A_x\ (1,0) + A_y\ (0,1) = A_x\ \hat{\imath} + A_y\ \hat{\jmath}$$

Can separate into two ⊥ vectors that add up to original, e.g.

$$\vec{A}_x = A_x\ \hat{\imath}, \quad \vec{A}_y = A_y\ \hat{\jmath}$$

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

Scalar product ("dot product") is a kind of multiplication that accounts for how well the two vectors are aligned with each other:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = |\vec{A}|\ |\vec{B}|\ \cos(\theta_{AB})$$

In one dimension, we learned

$$W = F_x\ \Delta x \quad \rightarrow \quad \int F_x(x)\ \mathrm{d}x$$

Sometimes the force is not parallel to the displacement: for instance, work done by gravity if you slide down a hill. In two dimensions,

$$W = \vec{F} \cdot \vec{D} = F_x\ \Delta x + F_y\ \Delta y$$

which in the limit of many infinitessimal steps becomes

$$W = \int \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} = \int (F_x(x,y)\ \mathrm{d}x + F_y(x,y)\ \mathrm{d}y)$$

Similarly, in two dimensions, power must account for the possibility that force and velocity are not perfectly aligned:

$$P = \vec{F} \cdot \vec{v}$$

Static friction and kinetic (sometimes called "sliding") friction:

$$F^{\text{Static}} \le \mu_S\ F^{\text{Normal}}$$

$$F^{\text{Kinetic}} = \mu_K\ F^{\text{Normal}}$$

"normal" & "tangential" components are $\perp$ to and $\parallel$ to surface.

For an inclined plane making an angle θ w.r.t. the horizontal, the normal component of gravity is $F^N = mg\cos\theta$ and the (downhill) tangential component is $mg\sin\theta$. The frictional force on a block sliding down the surface then has magnitude $\mu_K\ mg\cos\theta$ and points uphill if the block is sliding downhill. You have to think about whether things are moving and if so which way they are moving in order to decide which direction friction points and whether the friction is static or kinetic.

(Chapter 11: motion in a circle)

For motion in a circle, acceleration has a *centripetal* component that is perpendicular to velocity and points toward the center of rotation. If we put the center of rotation at the origin $(0,0)$ then

$$x = R\cos\theta \qquad y = R\sin\theta$$

$$\vec{r} = (x,y) = (R\cos\theta, R\sin\theta) = R\,(\cos\theta, \sin\theta)$$

The "angular velocity" ω is the rate of change of the angle θ

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

The units for ω are just s^{-1} (which is the same as radians/second, since radians are dimensionless). Revolutions per second are $\omega/(2\pi)$, and the period (how long it takes to go around the circle) is $2\pi/\omega$. The velocity is

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \omega R\,(-\sin\theta, \cos\theta), \qquad |\vec{v}| = \omega R$$

The magnitude of the centripetal acceleration (the required rate of change of the velocity vector, to keep the object on a circular path) is

$$a_c = \omega^2 R = \frac{v^2}{R}$$

and the centripetal force (directed toward center of rotation) is

$$|\vec{F}_c| = m a_c = m\omega^2 R = \frac{mv^2}{R}$$

Moving in a circle at constant speed (velocity changes but speed does not!) is called *uniform circular motion*. For UCM, $\vec{a} \perp \vec{v}$, and $\omega =$ constant. Then

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\omega^2 R\,(\cos\theta, \sin\theta) = -\frac{v^2}{R}\,(\cos\theta, \sin\theta)$$

(For non-UCM case where speed is not constant, $\vec{a}$ has an additional component that is parallel to $\vec{v}$.)

We can also consider circular motion with non-constant speed, just as we considered linear motion with non-constant speed. Then we introduce the *angular acceleration*

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

and we can derive results that look familiar but with substitutions

$$x \to \theta, \quad v \to \omega, \quad a \to \alpha, \quad m \to I, \quad p \to L$$

if α is constant (which is a common case for constant torque), then:

$$\theta_f = \theta_i + \omega t + \frac{1}{2}\alpha t^2$$

$$\omega_f = \omega_i + \alpha t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\ (\theta_f - \theta_i)$$

Rotational inertia ("moment of inertia") (see table below):

$$I = \sum mr^2 \quad \rightarrow \quad \int r^2 \mathrm{d}m$$

Kinetic energy has both translational and rotational parts:

$$K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2$$

Angular momentum:

$$L = I\omega = m\ v_\perp\ r_\perp$$

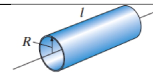
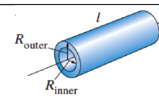
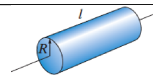
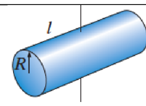
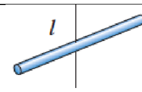
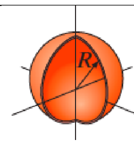
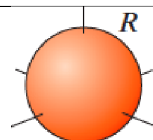
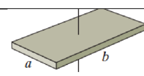
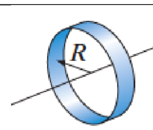
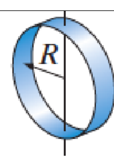
(where $\perp$ means the component that does not point toward the "reference" axis — which usually is the rotation axis)

If an object revolves about an axis that does not pass through the object's center of mass (suppose axis has $\perp$ distance ℓ from c.o.m.), the rotational inertia is larger, because the object's c.o.m. revolves around a circle of radius ℓ and in addition the object rotates about its own center of mass. This larger rotational inertia is given by the *parallel axis theorem:*

$$I = I_{\mathrm{cm}} + M\ell^2$$

where I_{cm} is the object's rotational inertia about an axis (which must be parallel to the new axis of rotation) that passes through the object's c.o.m.

(Chapter 12: torque)

configuration		rotational inertia
thin cylindrical shell about its axis		mR^2
thick cylindrical shell about its axis		$(1/2)m(R_i^2 + R_o^2)$
solid cylinder about its axis		$(1/2)mR^2$
solid cylinder $\perp$ to axis		$(1/4)mR^2 + (1/12)m\ell^2$
thin rod $\perp$ to axis		$(1/12)m\ell^2$
hollow sphere		$(2/3)mR^2$
solid sphere		$(2/5)mR^2$
rectangular plate		$(1/12)m(a^2 + b^2)$
thin hoop about its axis		mR^2
thin hoop $\perp$ to axis		$(1/2)mR^2$

Torque:

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$$

$$\tau = I\alpha$$

Work and power:

$$W = \tau\ (\theta_f - \theta_i)$$

$$P = \tau\omega$$

Equilibrium:

$$\sum \vec{F} = 0, \quad \sum \vec{\tau} = 0$$

(Chapter G9: static equilibrium, etc.)

Static equilibrium (all forces/torques acting ON the object sum to zero):

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum \tau = 0$$

Young's modulus: $\frac{\Delta L}{L_0} = \frac{1}{E}\ \left(\frac{\text{force}}{\text{area}}\right)$

(Onouye/Kane ch1: introduction)

static loads: gravitational forces, due to the weight of the structure or its contents. Includes *dead loads* due to the weight of the building and permanently attached components thereof, and *live loads* that come and go, such as furniture and people.

dynamic loads: inertial forces, due to resisting the motion of mass. For example: wind, vibration, earthquakes, falling objects.

(Onouye/Kane ch2: statics)

A force is characterized by *point of application*, magnitude, and direction. The force's *line of action* passes through the point of application of the force and is in the same direction as the force.

When we idealize an object as a *rigid body*, we assume that it undergoes negligible deformation in response to applied forces. If you think of a body as a huge number of constituent particles, the body is rigid if the relative distance between every pair of consituent particles is fixed. The rules of *statics* (i.e. static equilibrium) apply to rigid bodies. Statics in a plane gives us 3 equations (see below), allowing us to solve for 3 unknown forces (or 3 unknown force components, if directions are unknown). If there are more than 3 unknown force components, then we need additional information about how the body deforms in response to applied forces (i.e. the rigid-body idealization is no longer sufficient); that goes beyond the scope of statics. [But occasionally one can use statics to determine more than 3 unknown forces by using e.g. mirror symmetry to eliminate all but 3 unknowns.]

Usually a *load* is a specified external force that a structure must be designed to bear, such as the weight of snow on the roof or the weight of the building itself. Usually a *reaction* is an unknown external force whose value is calculated by imposing the conditions of static equilibrium on an object. If you and I sit on a see-saw, our weights and the weight of the wooden plank are loads; the upward contact force exerted by the pin on the center of the plank is a reaction. A free-body diagram for the plank includes the specified loads and the to-be-determined reaction.

Principle of transmissibility (applies to rigid bodies only): the acceleration and angular acceleration of a rigid body are unchanged by replacing a given force F_1 acting at point A with a new force F_2 acting at point B as long as forces A and B have the same line of action and point in the same direction.

Concurrent forces have lines of action that intersect at a common point. The effects on a rigid body are unchanged by replacing several concurrent forces with a single resultant force. The *resultant* of several forces is the vector sum of those forces.

The *moment* of a force is engineers' term for what physicists call torque. It is force multiplied by perpendicular lever arm, with a sign given by the convention that counterclockwise is positive and clockwise is negative. In 3 dimensions, a torque (or moment) is given by the vector ("cross") product $\vec{\tau} = \vec{r} \times \vec{F}$ and the right-hand rule. You can't define a moment (torque) without first defining a reference point, also known as a pivot, or an axis, or an origin for a coordinate system. The vector $\vec{r}$ in the expression $\vec{r} \times \vec{F}$ is measured with respect to that pivot point, i.e. the tail of $\vec{r}$ is at the pivot.

Varignon's theorem: to compute the moment of a force, you can decompose the force into components (having the same point of application) and sum (algebraically, i.e. with proper signs) the moments of the components.

A *couple* is two forces that sum to zero ($\vec{F}$ and $-\vec{F}$) and have parallel (you might say antiparallel) lines of action separated by a distance d. A couple will tend to cause rotational acceleration but will not cause linear acceleration of a body. The moment of a couple has magnitude Fd.

A force $\vec{F}$ acting on a rigid body can be moved to any given point of application A (with a parallel line of action) provided that a couple $\vec{M}$ is added. The moment M of the couple equals $Fd_\perp$, where $d_\perp$ is the perpendicular distance between the original line of action and the new location A.

In the 2D plane, the three equations of statics are: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum_{\circlearrowleft P} M = 0$, where P is a chosen pivot point for evaluating moments.

When engineers and architects say *Free Body Diagram*, they are referring to what

Mazur calls an *Extended Free Body Diagram.* An EFBD starts with a cartoon-like sketch of the body in question and indicates with an arrow each external force acting on the body, carefully indicating the direction and the point of application of the force. Often unknown reaction forces are drawn with a single slash through the arrow. External moments (illustrated via types of connections) are indicated using curved arrows. An unknown moment reaction is indicated using a single slash through a curved arrow.

Support forces are often drawn as stereotyped *pin* (or hinge) supports and *roller* supports. A pin can exert both horizontal and vertical support (reaction) forces but cannot exert any moment (torque) about the pin axis. A roller can only exert a force normal to the surface on which it rolls and cannot exert a moment. So a pin (or hinge) support contributes two unknown reaction force components, while a roller support contributes only one unknown reaction force component. A body that has a pin support beneath one end and a roller support benath the opposite end is *simply supported.* One pin and one roller support constitute 3 total unknown forces, which is exactly the number of unknowns that the laws of statics in a 2D plane can determine.

Another type of connection, not illustrated in chapter 2, is a "built-in" connection, which (in the 2D plane) can exert two forces and a moment. For an example, think of how a lamppost is attached to the sidewalk. It resists motion along its axis, resists motion parallel to the sidewalk, and also resists the toppling over of the lamppost, i.e. it resists rotation about the point of connection.

A body on which more than three unknown forces are exerted is called *statically indeterminate.* To solve a statically indeterminate system, you need to know how the body deforms under the applied load.

(Onouye/Kane ch3: selected determinate systems)

The resultant force exerted on the end of a cable must be tangent to the end of the cable.

A *concentrated load* has a point of application that can be represented as a single point. For example, the weight of a single lead brick placed at the center of a long beam. A *distributed load* is spread out over a wide area (usually indicated as force per unit length on a 2D sketch). The most common symbol used for concentrated (or "point") loads is P. The most common symbol used for distributed loads is w, though some books use ω.

For statics calculations of rigid bodies (but not for elastic calculations such as the deflection of beams!) a distributed load w can be replaced by the equivalent concentrated load P. The point of application of P is the centroid of w, and the magnitude of P is the integral of w, i.e. the area under the $w(x)$ curve, $\int w(x)\,\mathrm{d}x$. Usually this "integral" can be simply calculated using formulas for the area of a rectangle, a triangle, a trapezoid, etc.

There are two ways to analyze a *truss*: one is the *method of joints* and the other is the *method of sections*. Analysis of a truss assumes: (a) members (*bars*) are straight line segments and can support only axial forces, i.e. forces parallel to the axis of the bar; (b) all *joints* are pin connections, i.e. connections that can exert horizontal and vertical forces but not moments about the pin; (c) the weight of the truss bars themselves is usually neglected; (d) loads are applied to the truss at the pinned joints only. A given bar is either in *compression* (the forces exerted on the ends of the bar are trying to squish the bar along its axis) or in *tension* (the forces exerted on the ends of the bar are trying to stretch the bar along its axis).

A necessary condition for a planar truss having J joints and B bars to be solvable using the methods of statics is $B = 2J - 3$. Solving the truss involves finding B unknown bar tensions/compressions plus 3 unknown support reactions (e.g. one pin and one roller support). The method of joints will give us 2 equations per joint. So we have $2J$ equations to determine $B + 3$ unknowns. Thus $2J = B + 3$.

The *method of joints* is conceptually simple, but can be tedious. At each joint, you apply the two force equations for static equilibrium: $\sum F_x = 0$ and $\sum F_y = 0$ (consider forces acting *on the joint* itself). There is no moment equation because all forces at the joint have lines of action passing through the joint. I usually label the support "reaction" forces e.g. R_{Ax}, R_{Ay}, R_{Cy} for reaction forces at joints A and C, and then label the tension/compression of each bar as if every bar were in tension: T_{AB}, T_{BC}, T_{AC} for bars AB, BC, AC connecting joints A, B, C. In the end, you will find $T_{AB} > 0$ if bar AB is in tension and you will find $T_{AB} < 0$ if bar AB is in compression. To eliminate the need to solve large systems of simultaneous equations, always start from a joint having at most two unknown forces; if you find a joint having only one unknown force, so much the better.

In the *method of sections* you often (but not always) start by drawing an EFBD for the truss as a whole and solving for the unknown support reactions; sometimes this step is unnecessary. Then you draw a hypothetical line (or curve) that divides the truss into two pieces; the line should pass through bars, not joints, and should cut through no more than three bars whose forces are unknown. If there is a particular bar whose tension/compression you want to find, be sure that your cut line passes through that bar. You then draw an EFBD for *either* the right side or the left side of the truss, including the forces exerted (by the invisible side of the truss) on the bars cut by the section line. Be careful with the directions: if the cut bar (let's say it's bar AB) is assumed to be in tension, then the EFBD for the right side of the cut includes T_{AB} pointing (in general diagonally) to the left; alternatively the EFBD for the left side of the cut would include T_{AB} pointing (in general diagonally) to the right. You want to draw the external forces exerted **on** the part of the truss whose EFBD you have drawn. You then use the three equations for static equilibrium in a plane: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum_{\circlearrowleft P} M = 0$, where the pivot point P is strategically chosen so that the moment equation omits any forces that you do not care about. (Forces whose lines of action pass through the pivot P will have zero lever arm and will thus not appear in the moment equation.) You are summing forces and moments acting on the visible (i.e. left or right) portion of the truss as a whole. Whereas the method of joints found the conditions for each joint to be in equilibrium, the method of sections finds the conditions for the visible half of the truss as a whole to be in equilibrium. If you are only interested in finding a single bar force, and if you choose just the right section, and if you choose just the right pivot point, you can often find the desired force by solving only the moment equation. The method of joints is a brute-force method that you can imagine programming a computer to do for you; the method of sections requires some finesse.

Pinned frames and multiforce members are outside the scope of this course. Retaining walls are also outside the scope of this course.

(Onouye/Kane ch4: load tracing)

You work your way from the top to the bottom. The roof supports only its own weight. The top floor supports its own weight and the roof. The lower floor supports its own weight and everything above it. It's like finding all of the forces in a system of three blocks stacked one on top of the other: draw an FBD for the top block first. Then use what you know about the force exerted by the middle block on the top block (plus Newton's 3rd law) in drawing an FBD for the middle block. Then use what you know about the force exerted by the lower block on the middle block (plus the 3rd law) to draw an FBD for the lower block. (Notice that we usually use the opposite procedure when objects are suspended, in a chain, from the ceiling by cables. In that case it is easiest to work from the bottom up.)

The other key idea is *tributary area*: if the floor as a whole must support $2000\,\mathrm{N/m^2}$, and if the floor joists are spaced $0.5\,\mathrm{m}$ apart, then you attribute to each floor joist a distributed load $w = 1000\,\mathrm{N/m}$. Notice how we went from force per unit area to force per unit length.

(Onouye/Kane ch5: strength of materials)

Stress (usually denoted f in this book) is force per unit area. Strain (usually denoted ε in this book) is (change in length) divided by (original length). So ε is a dimensionless ratio. A graph of f vs ε shows many interesting features. One prominent feature is the *proportional* region in which $f = E\varepsilon$, a result called *Hooke's Law*, where E is called *Young's modulus* a.k.a. the elastic modulus or the modulus of elasticity. E is the slope of the straight-line portion of the f vs ε graph.

Ductile materials give warning of impending failure, while *brittle* materials do not.

If you stretch an iron rod along its axis, it will become thinner in the direction perpendicular (transverse) to its axis. So the axial strain is positive, but the transverse strain is negative. *Poisson's ratio* (symbol μ in this book, but ν is common elsewhere) is $\mu = -\varepsilon_{\text{transverse}}/\varepsilon_{\text{axial}}$. For most metals, $0.2 \le \mu \le 0.4$. So if you stretch a metal rod to be 1% longer axially, it will become about 0.3% thinner in the transverse direction.

(Onouye/Kane ch6: cross-sectional properties)

The *centroid*, denoted $(\overline{x}, \overline{y})$, is the mass-weighted average of the centers-of-mass of the constituent parts: $\overline{x} = (\sum_i x_i m_i)/(\sum_i m_i)$, and $\overline{y} = (\sum_i y_i m_i)/(\sum_i m_i)$, where m stands for mass. To find the centroid of a continuous object, use an integral instead of a sum: $\overline{x} = (\int x\,\mathrm{d}m)/(\int \mathrm{d}m)$, and $\overline{y} = (\int y\,\mathrm{d}m)/(\int \mathrm{d}m)$. If the material is of uniform density and thickness, then you can use area A instead of mass m.

The centroid of a right triangle one side of which lies along the x axis (base b) and one side of which lies along the y axis (height h) is $(\overline{x}, \overline{y}) = (b/3, h/3)$. The area is of course $bh/2$.

If a shape has a hole in it, you can "subtract" the hole from the shape by using a negative area for the hole in the centroid calculation!

The *second moment of area* (which this book calls *moment of inertia of an area,* and most engineers and architects simply call *moment of inertia*) is most commonly given by $I_x = \int y^2\,\mathrm{d}A$. Second moment of area is a difficult but important concept that helps to explain why an I-beam has the shape it has (with material far away from the $y = 0$ plane) and why a floor joist ("on edge") is stiff but the same board used as a plank ("on the flat") is floppy. As we'll see, a larger I_x makes a beam more stiff.

I avoid the phrase "moment of inertia" because it is ambiguous: most structures books use the phrase "moment of inertia" to refer to what I call "second moment of area," while most physics books use the phrase "moment of inertia" to refer to what Mazur calls "rotational inertia." Saying "rotational inertia" and "second moment of area" is always unambiguous, while saying "moment of inertia" is often ambiguous.

For a beam of rectangular cross-section $b \times h$ and uniform density supporting a vertical load, $I_x = bh^3/12$. Imagine a wooden beam (like a "two by ten") whose cross-section has small dimension d and large dimension D (e.g. maybe $d = 4\,\mathrm{cm}$ and $D = 20\,\mathrm{cm}$). If you orient the beam "on edge", like this ▯, then you get $I_x($▯$) = dD^3/12$. If you orient the beam "on the flat," like this ▭, then you get $I_x($▭$) = Dd^3/12$. The ratio is $I_x($▯$)/I_x($▭$) = (D/d)^2$, which is $5^2 = 25$ for the numbers given above. So the same piece of wood is $25\times$ stiffer (for these example numbers) when oriented as a joist than it is when oriented as a plank.

I_x, which represents how far the material of a beam is spread out from the $y = 0$ plane, is called I_x because if you draw a cross-section of the beam, the $y = 0$ plane is the x axis. So in cross-secton, I_x quantifies how far the material is from the x axis.

If a beam's cross-section consists of several components having cross-sectional areas A_1, A_2, A_3, vertical centroids y_1, y_2, y_3, and their own second moments of area I_{x1}, I_{x2}, I_{x3}, then you can compute the second moment of area of the composite beam using the *parallel axis theorem*: $I_x = I_{x1} + I_{x2} + I_{x3} + A_1 y_1^2 + A_2 y_2^2 + A_3 y_3^2$. You could use this, for example, to find I_x for an I-beam: ⌶ . Using more general notation, the parallel axis theorem reads $I_x = \sum_i (I_{xi} + A_i y_i^2)$. Warning: the way I've written this expression, you must choose $y = 0$ to be the vertical centroid of the cross-section, i.e. you must ensure that $\overline{y} = (\sum_i y_i A_i)/(\sum_i A_i) = 0$.

The *radius of gyration* $r_x = \sqrt{I_x/A}$ is the distance from the x axis at which you could concentrate all of the beam's material (symmetrically above and below) to get the same second moment of area I_x. Notice that $I_x = Ar_x^2$. The only place you are likely to use the radius of gyration is in calculating a *slenderness ratio* of a column. (Reading O/K ch9 on columns is an extra-credit option.)

(Onouye/Kane ch7: simple beams)

The most common support configurations for beams are *simply supported* (pin beneath one end and roller beneath the other end), *overhang* (like simply supported, but ends of beam extend beyond one or both supports), and *cantilever* (one "fixed" end, and one "free" end).

A *load diagram* is basically an EFBD of the beam. Remember to include the vertical reaction forces exerted by the supports on the beam. Sometimes the load diagram is represented as a graph of the distributed load $w(x)$ (force per unit length). In load, shear, and moment diagrams, the coordinate x measures distance along the length of the beam, starting from the left end of the beam. It is confusing that this meaning of the coordinate x is different from its meaning in chapter 6 — at least y will have the same meaning here as in chapter 6.

The *shear diagram*, $V(x)$, is drawn directly below the load diagram. $V(x)$ has dimensions of force (newtons, kilonewtons, pounds, kilopounds ("kips")). If you section the beam into two halves at a distance x from the left end of the beam, the function $V(x)$ represents the upward force exerted by the left side on the right side of the beam at that section. Equivalently, $V(x) = -\int w(x)\,\mathrm{d}x$. Also equivalently, $V(x)$ is the running sum of the loads and reactions (upward minus downward) to the left of (and including) the section at x.

The *moment diagram*, $M(x)$, is drawn directly below the shear diagram. $M(x)$ has dimensions of a bending moment (or torque), i.e. force×distance (newton-meters, kilonewton-meters, foot-pounds, kilopound-feet). If you section the beam into two halves at a distance x from the left end of the beam, the absolute value of the function $M(x)$ represents the magnitude of the moment (torque) that one side of the beam exerts on the other side. But the sign convention is such that "a positive moment makes the beam smile." So if the beam curves upward (smiles) under load (if $d^2Y/dx^2 > 0$) then $M > 0$, and if the beam curves downward (frowns) under load (if $d^2Y/dx^2 < 0$) then $M < 0$. Mathematically $M(x) = \int V(x)\,\mathrm{d}x$.

Since derivatives are less tricky than integrals, it may be worth remembering that $\mathrm{d}M(x)/\mathrm{d}x = V(x)$. The shear diagram $V(x)$ is the derivative (the slope) of the moment diagram $M(x)$. For distributed loads, it is also worth remembering that $\mathrm{d}V(x)/\mathrm{d}x = -w(x)$. The distributed load $w(x)$ is minus the slope of the shear diagram $V(x)$.

A free end, a pin-supported end, and a roller-supported end are all incapable of supporting a bending moment. So for any of those end conditions, $M(0) = 0$ and $M(L) = 0$. An exception is the cantilever beam, which has one free end and one fixed end. The fixed end of a cantilever has $M \neq 0$. Since a cantilever always frowns under a gravitational load, the fixed end has $M < 0$.

Sometimes one draws two additional curves beneath $M(x)$. The slope of the loaded beam, $\theta(x) = dY/dx$, is given by $EI\,\theta(x) = \int M(x)\,\mathrm{d}x$, where E is Young's modulus (elastic modulus) and I is the second moment of area (called I_x in chapter 6). If one draws $\theta(x)$, it is drawn directly below the $M(x)$ diagram. The deflected shape of the loaded beam, $Y(x)$ is given by $Y(x) = \int \theta(x)\, \mathrm{d}x$. If one draws $Y(x)$, it is drawn directly below the $\theta(x)$ diagram.

While you will probably never actually draw the $\theta(x)$ and $Y(x)$ curves, a key takeaway is that you integrate the $M(x)$ curve two more times to get $Y(x)$. That implies that if $M(x)$ is linear (a polynomial of order one), then the shape $Y(x)$ of the deflected beam is a polynomial of order three. And if $M(x)$ is quadratic (a polynomial of order two) then the shape $Y(x)$ of the deflected beam is a polynomial of order four. So it turns out that the maximum deflection of a beam of length L under a concentrated load is usually proportional to L^3, and the maximum deflection of a beam of length L under a uniform distributed load is usually proportional to L^4, just because of calculus.

(Onouye/Kane ch8: bending and shear stresses in beams)

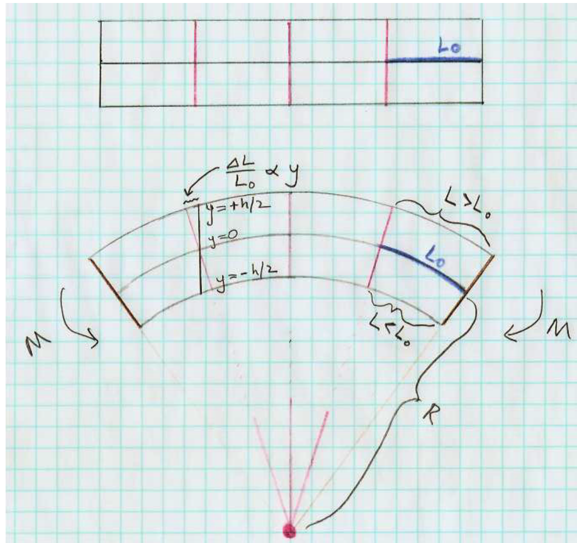
The *neutral axis* of a beam's cross-section lies along the vertical centroid $\overline{y}$ of the cross-section. Extending the neutral axis along the length of the beam defines the *neutral surface*. If certain conditions are met (the beam is initially straight, is of constant cross-section, and is of uniform composition; the beam is elastic and has equal elastic moduli in tension and compression; the beam is bent only with couples (bending moments at the ends); the beam is not twisted), then the longitudinal elements (fibers — easy to imagine for a wooden beam) of the neutral surface will be neither in tension nor in compression; they will undergo no change in length.

For a "simply supported" beam (which makes a $\smile$ shape under load), longitudinal fibers below the neutral surface are in tension (elongated), while fibers above the neutral surface are in compression (shortened). For a cantilever (which makes a $\frown$ shape under load), fibers above the neutral surface are in tension, while fibers below the neutral surface are in compression. It helps to imagine a wooden beam with fibers (grains) running along the axial length of the beam.

Let's imagine an initially horizontal beam of length L_0 bent into a $\frown$ shape by applying a bending moment M at each end: counterclockwise at the left end and clockwise at the right end. Fibers above the neutral axis ($y > 0$) will be lengthened ($L > L_0$) while fibers below the neutral axis ($y < 0$) will be shortened ($L < L_0$).

A key idea is that we can approximate the deflected beam as an arc of a circle of radius R, where the bending moment is inversely related to the radius of curvature of the beam: $M \propto 1/R$. The larger the bending moment, the tighter the circular arc into which the beam bends. For a constant bending moment M, lines that are initially vertical converge toward the center of the circle, as shown below.

We can use similar triangles to argue that the longitudinal strain, $\delta L/L_0$, is proportional to the distance above the neutral surface. More precisely, $\Delta L/L_0 = y/R$.



Next, use the symbol e to denote the axial strain $\Delta L/L_0$, and use the symbol f to denote stress, which is force/area. For an elastic material, $f = eE$, where E is Young's modulus. So we have $y/R = \Delta L/L_0 = e = f/E$. So the axial stress (force per unit area) exerted by the fibers a distance y above the neutral axis is $f = Ey/R$.

Now, using the language of calculus, consider an infinitesimal fiber of area $\mathrm{d}A$ a distance y above the neutral surface. Using a pivot along the neutral axis, the torque (bending moment) exerted by the longitudinal fiber of area $\mathrm{d}A$ equals force times lever arm. The force is $\mathrm{d}F = f\mathrm{d}A$ (stress times area) and the lever arm is y. So the infinitesimal bending moment exerted by this infinitesimal fiber is

$$\mathrm{d}M = y\,\mathrm{d}F = y\,f\mathrm{d}A = y\left(\frac{Ey}{R}\right)\mathrm{d}A = \frac{E}{R}\,y^2\,\mathrm{d}A.$$

So the bending moment M exerted by a curved beam is

$$M = \int \mathrm{d}M = \frac{E}{R}\int y^2\,\mathrm{d}A = \frac{EI}{R} \qquad (1)$$

where R is the curved beam's radius of curvature, and $I = \int y^2\,\mathrm{d}A$ is the "second moment of area" introduced in chapter 6.

Using $f = Ey/R$ to eliminate R, we can also write

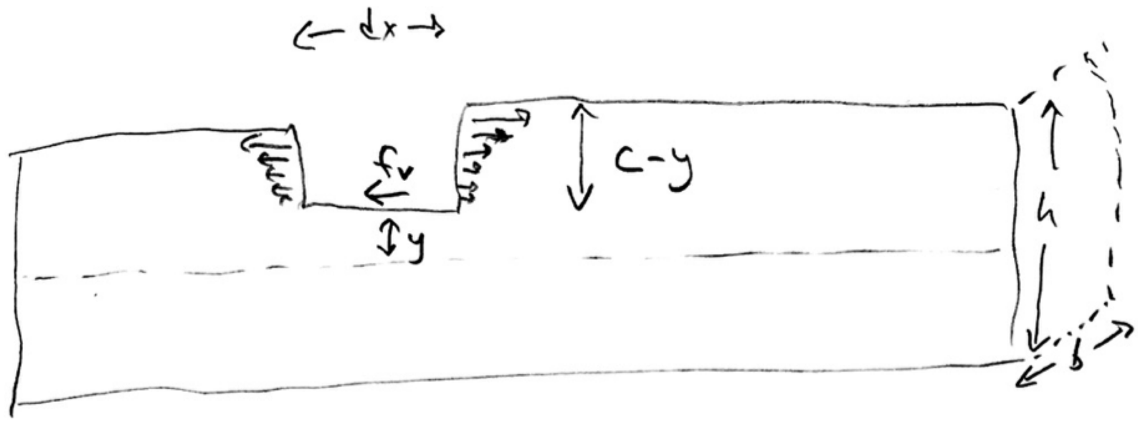
$$f = My/I, \qquad (2)$$

or using c to represent the most extreme value of $|y|$ (for the fibers farthest from the neutral surface), the maximum bending stress is $\boxed{f_{\max} = Mc/I}$. After drawing the moment diagram $M(x)$, you can use the maximum value of $|M(x)|$ and your knowledge of the beam cross-section to determine the maximum bending stress $f_{\max}$, which can be compared with the allowable stress F_{allow} for the material of which the beam is composed.

Since I and c are just geometrical properties of the beam cross-section, their ratio is given a name: $\boxed{S = I/c}$ is called the *section modulus*, where I is second moment of area (w.r.t. the neutral axis) and c is the distance from the neutral surface to the top or bottom of the beam (whichever is larger, if the beam is asymmetric). We can then write the bending stress in the extreme fibers as $\boxed{f_b = M/S}$. Alternatively, if you are working with material of a given allowable bending stress F_b and the maximum (in absolute value) bending moment for your loading conditions is $M_{\max}$, then you need to choose a cross-section for your beam whose section modulus is larger than $\boxed{S_{\text{required}} = M_{\max}/F_b}$. For standard beam shapes, values of section modulus S are tabulated. The dimensions of section modulus are length^3, e.g. cubic meters, cubic centimeters, or cubic inches.

(Need diagram.) Imagine a fiber located a height y above the neutral surface. At position x along the length of the beam, the axial (bending) stress in this fiber will be $f_b = My/I$, using equation (2). Because $M(x)$ varies along the length of the beam, this bending stress will vary with x:

$$\frac{\mathrm{d}f_b}{\mathrm{d}x} = \frac{y}{I}\frac{\mathrm{d}M(x)}{\mathrm{d}x} = \frac{y}{I}V(x)$$



(Need a nicer diagram.) Now imagine the forces acting on a rectangular block of beam that extends longitudinally from x to $x + \mathrm{d}x$, extends vertically from y to c (measured from the neutral surface, where c is the top surface of the beam), and extends the entire width b of the beam cross-section. Since stress = force/area, each force is the integral of stress over the corresponding area. The horizontal force acting on the left surface of the block is $\int_y^c f_b(x)\, b\, \mathrm{d}y$. The horizontal force acting on the right surface of the block is $\int_y^c f_b(x+\mathrm{d}x)\,b\,\mathrm{d}y$. Along the top surface there is no force, as there is no material above the top of the beam. But acting horizontally along the bottom surface of the rectangular block is the *shear stress*, f_v. The corresponding force is $f_v\,b\,\mathrm{d}x$. The horizontal forces on these three surfaces must sum arithmetically to zero:

$$f_v\,b\,\mathrm{d}x = \int_y^c [f_b(x+\mathrm{d}x) - f_b(x)]\,b\,\mathrm{d}y = \int_y^c [\frac{\mathrm{d}f_b}{\mathrm{d}x}\,\mathrm{d}x]\,b\,\mathrm{d}y = \int_y^c [\frac{y}{I}\,V(x)\,\mathrm{d}x]\,b\,\mathrm{d}y.$$

We can cancel $\mathrm{d}x$, and for the special case of a rectangular cross-section (so b is independent of y) we can cancel b, replace c with $h/2$, and replace I with $bh^3/12$:

$$f_v = \frac{V(x)}{I}\int_y^{h/2} y\,\mathrm{d}y = \frac{V}{I}\left[\frac{h^2}{8} - \frac{y^2}{2}\right] = \frac{12V}{bh^3}\left[\frac{h^2}{8} - \frac{y^2}{2}\right] = \frac{3V}{2A}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

where $A = bh$ is the area of the beam cross-section. The maximum shear stress is $\frac{3}{2}V/A$ (for a rectangular cross-section) and occurs at the neutral surface ($y = 0$) at the longitudinal position x where the shear force $|V(x)|$ is largest — which usually occurs at the supports.

To envision shear strain (which by Hooke's law is proportional to shear stress), bend a deck of cards into a shape and observe how each card slides against its neighbors.

In many circumstances, building codes will specify the maximum allowable deflection of a beam of length L as some small fraction of the length of the beam: for example, an $L/360$ deflection limit would imply that a horizontal beam of length 3.6 m can deflect no more than 1 cm vertically under load. We use the symbol Δ to indicate the vertical deflection of the beam. A positive value of Δ points downward, in the $-y$ direction. We can consider the deflection $\Delta(x)$ as a function of horizontal position x along the length of the beam, or we can consider the maximum deflection Δ_{max}. We want to be able to evaluate Δ_{max} for a hypothetical beam under load and impose an allowable deflection criterion, for example $\Delta_{\mathrm{max}} \le L/360$.

Solving equation (1) for R, the radius of curvature of a loaded beam is $R = EI/M$. The beam is straighter (larger R) when the elastic modulus E and second moment of area I are larger; the beam curves more (smaller R) when the bending moment M is larger. The radius of curvature R of a function $y = f(x)$ is given in calculus by the formula

$$\frac{1}{R} = \frac{y''}{(1+(y')^2)^{3/2}} \approx y''.$$

We know that the second derivative of a function is related to its curvature: if $y'' = 0$ then the function is a straight line (no curvature); if $y'' > 0$ then the function has "concave up" curvature; and if $y'' < 0$ then the function has "concave down" curvature. In architectural structures, one deals with beams whose slope is very small: $|y'| \ll 1$, meaning that the slope of the beam under load is much smaller than one radian. (A radian is $57.3°$, which would be a very large slope for a deflected beam.) So it is conventional to use the small-angle ($|y'| \ll 1$) approximation: $y'' \approx 1/R$.

In the small-angle approximation, the second derivative $\Delta''(x)$ of the deflected beam shape $\Delta(x)$ obeys the *Euler-Bernoulli beam equation*

$$-\Delta''(x) = \frac{1}{R} = \frac{M}{EI}.$$

The minus sign is because $\Delta(x)$ increases in the $-y$ direction. We can integrate the bending-moment curve $M(x)$ twice to get the deflected shape $\Delta(x)$ of the beam:

$$-\Delta(x) = \frac{1}{EI} \int \mathrm{d}x \int M(x)\,\mathrm{d}x$$

Since the moment curve $M(x)$ is usually a quadratic curve for a beam with a uniform distributed load w and is usually a piecewise linear curve for a beam with a concentrated load P, it makes sense that $\Delta(x)$ is usually a fourth-order polynomial for a uniformly loaded beam and is usually a cubic polynomial for a concentrated load. The most common cases are tabulated in books and online references. For example, a simply supported beam has $\Delta_{\max} = 5wL^4/(384EI)$ for uniform load w or $\Delta_{\max} = PL^3/(48EI)$ for a concentrated load P at mid-span. A cantilever has $\Delta_{\max} = wL^4/(8EI)$ for uniform load w and $\Delta_{\max} = PL^3/(3EI)$ for concentrated load P at the free end. You can look up many more specific cases.

Here's where the crazy 5/384 comes from: A simply supported beam of length L and uniform load w has shear curve $V(x) = (\frac{1}{2}L - x)w$ and bending moment curve $M(x) = (Lx - x^2)w/2$. Integrating twice,

$$\Delta(x) = -\frac{1}{EI} \int \mathrm{d}x \int M(x)\,\mathrm{d}x = -\frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12} + C_1 x + C_2\right)$$

The boundary condition $\Delta(0) = 0$ gives $C_2 = 0$ and $\Delta(L) = 0$ gives $C_1 = -L^3/12$. So $\Delta(x) = \frac{w}{2EI}\left(\frac{x^4}{12} - \frac{Lx^3}{6} + \frac{L^3 x}{12}\right)$. Plugging in $x = L/2$ (which is where $\Delta'(x) = 0$) gives $\Delta_{\max} = 5wL^4/(384EI)$. To get the two integration constants for a simply supported beam, use $\Delta(0) = \Delta(L) = 0$. For a cantilever whose left end is fixed, the integration constants would instead be given by $\Delta(0) = 0$ and $\Delta'(0) = 0$.

Beam design criteria usually include the following:

- Axial stress in the extreme fibers of the beam (farthest from the neutral surface) must be smaller than the allowable bending stress, F_b, which depends on the material (wood, steel, etc.) Maximum bending stress happens where bending moment $|M(x)|$ is largest.

- Shear stress, in both y ("transverse") and x ("longitudinal") directions, must be smaller than the allowable shear stress F_v, which also depends on the material (wood, steel, etc.). Shear stress is maximum where $|V(x)|$ is largest, and is largest near the neutral surface.

- The above two are "strength" criteria. A third condition is a "stiffness" criterion: The maximum deflection under load must satisfy the building code: typically $\Delta_{\max} < L/360$, though in some cases the denominator is smaller, e.g. 120, 180, 240. For a uniform load, the maximum deflection occurs farthest away from the supports. If deflection is too large, plaster ceilings develop cracks, and floors feel uncomfortably bouncy or sloped.

- Onouye/Kane also mention buckling as a beam failure mode. For a simply supported beam, the top is in compression while the bottom is in tension; vice-versa for a cantilever. In very deep beams (i.e. very tall in cross-section), the compression side can buckle or deflect sideways. Wood framing addresses this issue with sheathing (a.k.a. furring or strapping) nailed at close spacing perpendicular to the floor joists and solid blocking to prevent buckling at the ends. In a very deep I-beam, the flange on the compression side is susceptible to buckling.

(Onouye/Kane ch9: columns)

O/K ch9 is an extra-credit chapter. I will eventually summarize its key results here.

(Chapter 13: gravity) (Chapter 13 is optional/XC this year)

Gravity:

$$F = \frac{G m_1 m_2}{r^2}$$

where $\vec{F}$ points along the axis connecting m_1 to m_2.

$$G = 6.67 \times 10^{-11}\ \frac{\mathrm{N\ m^2}}{\mathrm{kg^2}}$$

is a *universal* constant — the same on Earth, on Mars, in distant galaxies, etc.

$$g = 9.8 \text{ m/s}^2 = \frac{GM_e}{R_e^2}$$

shows that an apple falling onto Newton's head results from the same force that governs the motion of the Moon around Earth, Earth around the Sun, etc.

For an orbit, gravity provides the centripetal force, so

$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

Gravitational potential energy for objects 1 and 2 is

$$U = -\frac{Gm_1m_2}{r} \quad \text{(note the sign)}$$

which $\to 0$ as $r \to \infty$. The objects are *bound* if $K + U < 0$.

If $K + U \geq 0$, they escape each other. They just barely escape if $K + U = 0$

$$\frac{1}{2}mv_{\text{escape}}^2 = \frac{GmM}{R}$$

in which case $K \to 0$ when $R \to \infty$.

G.P.E. of e.g. a spacecraft of mass m in the field of two large objects (e.g. earth and moon) of mass M_1 and M_2:

$$U = -\left(\frac{GM_1m}{R_{M1,m}} + \frac{GM_2m}{R_{M2,m}}\right)$$

For a central force that goes like $F \propto 1/R^2$, the forces from a uniform spherical shell add (if you're outside the shell) up to one force directed from the center of the shell. So a rigid sphere attracts you as if it were a point mass.

If you're inside the shell, the sum of the forces adds up to zero.

(Chapter 15: periodic motion)

Oscillations (mostly illustrate using mass and spring). Combining $F = ma$ with $F = -kx$, we get $m\ddot{x} = -kx$. (A dot is shorthand for derivative with respect to time.) Using $\omega = \sqrt{k/m}$, you can rewrite as $\ddot{x} = -\omega^2 x$, which has solution

$$x = A\cos(\omega t + \phi), \quad v_x = \dot{x} = -\omega A \sin(\omega t + \phi)$$

You can also write it in terms of frequency f, using $\omega = 2\pi f$:

$$x = A\cos(2\pi f\ t + \phi), \quad v_x = -2\pi f A \sin(2\pi f t + \phi)$$

where f is frequency (cycles per second) and ω is "angular frequency" (radians per second). When a frequency is given in Hz (hertz), it always means f, not ω. The A above middle C on a piano has frequency $f = 440$ Hz, and the buzzing you hear from electrical appliances is 60 Hz (or a small-integer multiple, e.g. 120 Hz).

So frequency is $f = \frac{\omega}{2\pi}$ (how many times the thing vibrates per second), period is $T = \frac{1}{f} = \frac{2\pi}{\omega}$ (how many seconds elapse per vibration). The maximum displacement is *amplitude* A, measured in meters. The maximum speed is ωA (units are meters/second). The initial phase, ϕ, tells you where you are in the oscillation at $t = 0$. If at $t = 0$ you have $x > 0$ but $v_x = 0$, then $\phi = 0$. If at $t = 0$ you have $v_x > 0$ but $x = 0$, then $\phi = \pi/2$ (90°). The energy is $K + U = \frac{1}{2}m\omega^2 A^2$.

For a pendulum, you get $\theta = A\cos(\omega t + \phi)$, with $\omega = \sqrt{g/\ell}$. This requires two approximations: first, that θ is small enough that $\sin\theta \approx \theta$; second, that the mass is concentrated at a point at the end of the string, i.e. that the shape of the mass does not contribute to the rotational inertia of the pendulum.

If the second approximation does not hold (e.g. the rod is about as heavy as the mass on the end), then you have a "physical pendulum" with $\omega = \sqrt{mg\ell_{cm}/I}$, where ℓ_{cm} is the distance from the pivot to the CoM, and I is the rotational inertia **about the pivot** (not about the CoM).

For damped oscillations, the *energy* decays away with a factor $e^{-t/\tau}$, where the symbol τ in this case means "decay time constant," (not torque!). The quality factor $Q = 2\pi f\tau$ tells you how many oscillation periods it takes for the oscillator to lose a substantial fraction ($1 - e^{-2\pi} \approx 99.8\%$) of its stored energy.

For two springs connected in parallel (side-by-side), $k = k_1 + k_2$. For two springs connected in series (end-to-end), $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$, or equivalently $k = \frac{k_1 k_2}{k_1 + k_2}$.

Waves (not until spring semester)

Wavelength λ, frequency f, and speed c of wave propagation are related by

$$c = \lambda f$$

For transverse waves on a taut string of mass per unit length m/L, speed c of wave propgation is

$$c = \sqrt{\frac{T}{m/L}}$$

Sound

For waves that spread out in three dimensions without reflection or absorption, intensity I at distance r is given in terms of source power P by

$$I = \frac{P}{4\pi r^2}$$

Intensity level, β, of sound (in decibels) is given by

$$\beta = 10 \text{ dB } \log_{10}\left(\frac{I}{I_0}\right)$$

where $I_0 = 10^{-12}\ \text{W/m}^2$ is the threshold of human hearing and $\log_{10}$ means taking the base-ten logarithm.

Speed of sound in air is $c_\text{sound} = \sqrt{B/\rho}$, or 343 m/s at 20°C and 331 m/s at 0°C, where B is the bulk modulus and ρ is the density (mass/volume).

For observer moving away from (toward) stationary source, Doppler-shifted frequency is (use upper sign for "away from" and lower sign for "toward")

$$f_\text{observed} = f_\text{emitted}\left(1 \mp \frac{v_\text{observer}}{c_\text{sound}}\right)$$

For source moving away from (toward) stationary observer,

$$f_\text{observed} = \frac{f_\text{emitted}}{1 \pm v_\text{source}/c_\text{sound}}$$

Angle of shock wave for sonic boom is given by $\sin\theta = c/v$.

Light

Angle of incidence (w.r.t. surface normal) equals angle of reflection. Incident ray, reflected ray, refracted ray, and surface normal all lie in a plane.

Speed of light in vacuum: $c = 299792458$ m/s $\approx 3.00 \times 10^8$ m/s. Speed of light is c/n in medium with index of refraction n.

For geometrical (ray) optics, light obeys *principle of least time,* which is also known as Fermat's principle.

Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Lens & mirror summary (light always enters from LHS):

converging lens	$f > 0$	$d_i > 0$ is RHS	$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$
diverging lens	$f < 0$	$d_i > 0$ is RHS	$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$
converging mirror	$f > 0$	$d_i > 0$ is LHS	$f = R/2$
diverging mirror	$f < 0$	$d_i > 0$ is LHS	$f = -R/2$

Horizontal locations of object, image (beware of sign conventions!):

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \qquad \Rightarrow \qquad d_i = \frac{d_o f}{d_o - f}$$

Magnification (image height / object height):

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} = \frac{f}{f - d_o}$$

Lenses: $R_{1,2} > 0$ for "outie" (convex), < 0 for "innie" (concave).

Real image: $d_i > 0$. Virtual image: $d_i < 0$. Real image means light really goes there. Virtual: rays converge where light doesn't go.

Lens maker's equation (usually $n_0 = 1$ for air; flat surface $R = \infty$):

$$\frac{1}{f} = \left(\frac{n}{n_0} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Focusing "power" (in diopters: $1\ \mathrm{D} = (1\ \mathrm{m})^{-1}$) for a lens is $1/f$.

Brewster's angle (reflected light is polarized if $\theta_i > \theta_B$): $\theta_B = \arctan(n_2/n_1)$

For a thin film surrounded by air (e.g. soap bubble), a film of thickness $\lambda/4n$ will have maximum reflection for normal incidence, where n is the film's index of refraction.

Rayleigh criterion: a lens (or other aperture) of diameter D can resolve angles no smaller than

$$\theta_{\min} = \frac{1.22\ \lambda}{D}$$

Visible light: $\lambda_{\text{red}} \approx 630$ nm, $\lambda_{\text{green}} \approx 540$ nm, $\lambda_{\text{blue}} \approx 450$ nm, $\lambda_{\text{violet}} \approx 380$ nm.

For a two-lens telescope, where the objective lens and the eyepiece have focal lengths f_o and f_e, respectively, the magnification is $M = -f_o/f_e$.

When monochromatic light passes through two narrow slits that are separated by distance δ, and is viewed on a screen at a large distance L, the separation Δx between adjacent maxima in the interference pattern is $\Delta x = \lambda L/\delta$. Equvalently, the angle θ between adjacent maxima is given by $\sin\theta = \lambda/\delta$. Notice that $\Delta x/L \approx \sin\theta$.

Fluids

pressure: $P = F/A$.   1 Pa = 1 N/m^2.   1 atm = 101325 Pa = 760 mm-Hg.

Pascal's principle: if an external pressure is applied to a confined fluid, the pressure at every point within the fluid increases by that amount.

Archimedes's principle: the buoyant force on an object immersed (or partially immersed) in a fluid equals the weight of the fluid displaced by that object.

Equation of continuity: $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

Bernoulli's equation (neglects viscosity, assumes constant density):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

Viscosity (symbol η, unit = Pa·s), where F is the frictional force between two parallel plates of area A, separated by distance d, moving at relative velocity v, is defined by

$$F = \frac{\eta A v}{d}$$

Reynolds number Re indicates presence of turbulence. For Re < 2300, flow is *laminar.* For Re > 4000, flow is *turbulent.* For 2300 < Re < 4000, turbulent flow is possible ("onset of turbulence"). For average fluid speed $\overline{v}$ in a cylinder of radius r,

$$\text{Re} = \frac{2r\rho\overline{v}}{\eta}$$

Poiseuille's equation for volume rate of flow Q (unit = m^3/s) for a viscous fluid undergoing laminar flow in a cylindrical tube of radius R, length L, end-to-end pressure difference $P_1 - P_2$, is

$$Q = \frac{\pi R^4 \ (P_1 - P_2)}{8\eta L}$$

Surface tension $\gamma = F/L$ is force per unit length, tending to pull the surface closed. You can also regard γ as the energy cost per unit increase in surface area.

Useful tables: densities (Giancoli Table 10-1, page 276); viscosities (Mazur Table 18-1; or Giancoli Table 10-3, page 295); surface tensions (Mazur Table 18-2; or Giancoli Table 10-4, page 297).

Kinetic theory, heat, thermodynamics

Atomic mass unit: 1 u = 1.66×10^{-27} kg. Proton mass: $m_p = 1.6726 \times 10^{-27}$ kg. Neutron mass: $m_n = 1.6749 \times 10^{-27}$ kg. As you saw if you did the extra-credit reading on Special Relativity (Einstein's $E = mc^2$, etc.), the mass of a nucleus is slightly smaller than the sum of the masses of its protons and neutrons, because of the negative binding energy that holds the nucleus together. So you can argue that mass is really just one more form of energy! The atomic mass unit u is defined to be $\frac{1}{12}$ of the mass of a ^{12}C nucleus, which is a bound state of 6 protons and 6 neutrons.

Avogadro's number: $N_A = 6.022 \times 10^{23}$. Just as 12 of something is called a dozen, 6.022×10^{23} of something is called a mole. The mass of a mole of protons is 1.007 grams, i.e. almost exactly a gram. A mole of atomic mass units is $N_A \times 1$ u $=$ 1.0000 g $= 1.0000 \times 10^{-3}$ kg.

A Fahrenheit degree is $\frac{5}{9}$ of a Celsius degree, and 0°C is 32°F. According to the Wikipedia, the Fahrenheit scale is considered obsolete everywhere except the United States, the Cayman Islands, and Belize.

The Kelvin scale measures absolute temperature. A change of one Kelvin is the same as a change of 1°C, but with an offset such that 0°C = 273.15 K.

Thermal expansion: $\Delta L = \alpha L_0 \Delta T$, $\Delta V = \beta V_0 \Delta V$. Typically $\beta = 3\alpha$. (Here α is the linear coefficient of thermal expansion, and β is the volume coefficient of thermal expansion.)

Thermal stress (if ends are not allowed to move when object is heated or cooled): $F/A = E\alpha\Delta T$

Ideal gas law (works where density is low enough that the gas molecules interact primarily with the walls of the container, and not so much with one another):

$$PV = nRT$$

where T is in Kelvin and n is in moles. If you measure P in Pa (same as N/m^2) and V in m^3, then $R = 8.315\ \frac{\mathrm{J}}{\mathrm{mol \cdot K}}$. If you measure P in atm and V in liters, then $R = 0.0821\ \frac{\mathrm{L \cdot atm}}{\mathrm{mol \cdot K}}$. A mol of ideal gas at STP (1 atm, 0°C) has a volume of 22.4 L.

The volume per mole of ideal gas at temperature T at 1 atm is

$$\frac{V}{n} = \frac{RT}{P} = \frac{(0.0821\ \frac{\mathrm{L \cdot atm}}{\mathrm{mol \cdot K}})T}{1\ \mathrm{atm}} = (22.4\ \mathrm{L})\left(\frac{T}{273\ \mathrm{K}}\right) = (0.0224\ \mathrm{m}^3)\left(\frac{T}{273\ \mathrm{K}}\right)$$

If you measure N in molecules (not moles), P in $\mathrm{N/m}^2$, V in m^3, and T in Kelvin, then $PV = Nk_BT$, where $k_B = 1.38 \times 10^{-23}$ J/K is **Boltzmann**'s constant. The root-mean-squared speed of a gas molecule, v_{rms} is given by $\frac{1}{2}mv_{\mathrm{rms}}^2 = \frac{3}{2}k_BT$, with T in Kelvin. So the average K.E. of a gas molecule is proportional to absolute temperature.

Because of the random motions of molecules, a concentrated blob of ink, spray of perfume, etc., will **diffuse** from a region of high concentration to a region of low concentration. The number per unit time of molecules diffusing through area A is $\frac{\mathrm{d}N}{\mathrm{d}t} = AD\frac{\mathrm{d}C}{\mathrm{d}x}$, where C is the concentration of molecules per unit volume, and D is called the diffusion constant (unit is m^2/s).

Increasing the temperature of a given mass of a given substance requires heat $Q = mc\Delta T$, where c is the **specific heat capacity** of the substance. (Mind the sign: you get heat back out if you decrease the temperature.)

Melting or evaporating a mass m of a substance requires heat $Q = mL$, where L is the **latent heat** (of fusion for melting, of vaporization for boiling). (Mind the sign: you get heat back out for condensation or for freezing.)

Remember that **energy is conserved** (always, now that we know how to account for thermal energy). Work W represents the transfer of mechanical energy into ($W > 0$) or out of ($W < 0$) a system. Heat Q represents the transfer of thermal energy into ($Q > 0$) or out of ($Q < 0$) a system. If we call the internal energy (including thermal energy) of the system U, then

$$\Delta U = W_{\mathrm{in}} + Q_{\mathrm{in}} - W_{\mathrm{out}} - Q_{\mathrm{out}}$$

is just the statement that energy is conserved.

Heat (symbol Q, standard (S.I.) unit = joules) is the transfer of thermal energy from a warmer object to a cooler object. Heat can be transferred via conduction, convection, and radiation.

Conduction is the incoherent movement (similar to diffusion) of thermal energy through a substance, from high T to low T. The heat conducted per unit time is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{k\ A\ \Delta T}{\ell} = \frac{A\ \Delta T}{R}$$

where k is thermal conductivity, A is cross-sectional area (perpendicular to direction of heat flow), ℓ is the thickness (parallel to direction of heat flow), and ΔT is the temperature difference across thickness ℓ. We can also define **R-value**, $\mathbf{R} = \ell/\mathbf{k}$, and then use the second form written above. Be careful: if an R-value is given in imperial units ($\text{foot}^2 \cdot \text{hour} \cdot {}^\circ\text{F/Btu}$), you must multiply it by 0.176 to get S.I. units ($\text{m}^2 \cdot \text{K/W}$).

Convection means e.g. I heat some water in a furnace, then a pump mechanically moves the hot water to a radiator: it is the transfer of thermal energy via the coherent movement of molecules. Convection also occurs if I heat some air, which then becomes less dense and rises (because of buoyancy, which is caused by gravity), moving the thermal energy upward. "Heat rises" because increasing temperature usually makes things less dense, hence more buoyant.

Radiation is the transfer of heat via electromagnetic waves (visible light, infrared, ultraviolet, etc.), which can propagate through empty space. For a body of emissivity e ($0 \le e \le 1$, 0 = shiny, 1 = black) at temperature T (kelvin), with surface area A, the heat radiated per unit time is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = e\sigma A T^4$$

where $\sigma = 5.67 \times 10^{-8}\ \frac{\text{W}}{\text{m}^2 \cdot K}$ is the Stefan-Boltzmann constant.

Useful tables: expansion coefficients (Giancoli Table 13-1, page 388); saturated vapor pressure of water (Giancoli Table 13-4, page 406); specific heat capacities (Giancoli Table 14-1, page 421); latent heats (Giancoli Table 14-3, page 425); thermal conductivities (Giancoli Table 14-4, page 429).

Useful tables for elasticity, etc.: elastic modulus (Giancoli Table 9-1, page 254); ultimate strength (Giancoli Table 9-2, page 258).

Work done BY a gas is $W_{\text{out}} = \int P\ \mathrm{d}V$. Work done ON a gas is $W_{\text{in}} = -\int P\ \mathrm{d}V$. Mazur's convention is $W = -\int P\ \mathrm{d}V$, i.e. if you don't specify "in" or "out" then W means $W_{\text{in}} - W_{\text{out}}$.

The state (P, V, T, S, energy) of a steady device is the same at the end of each cycle. Since the energy is unchanged after going around one complete cycle, all of the changes in energy must add up to zero, so then

$$W_{\text{in}} + Q_{\text{in}} = W_{\text{out}} + Q_{\text{out}}$$

If a system transfers thermal energy Q_{out} to its environment at constant temperature T_{out}, the change in the entropy of the environment is $\Delta S_{\text{env}} = Q_{\text{out}}/(k_B T_{\text{out}})$. (This is using Mazur's definition, $S = \ln \Omega$, for entropy, which is usually called the "statistical entropy." Most other books instead use $S = k_B \ln \Omega$, which is called the "thermodynamic entropy." Books that use $S = k_B \ln \Omega$ will instead write $\Delta S_{\text{env}} = Q_{\text{out}}/T_{\text{out}}$.)

If a system absorbs thermal energy Q_{in} from its environment at constant temperature T_{in}, the change in the entropy of the environment is $\Delta S_{\text{env}} = -Q_{\text{in}}/(k_B T_{\text{in}})$.

If thermal energy is transferred at non-constant temperature, you can use calculus to figure out $\Delta S = \int \frac{1}{T}\, \mathrm{d}Q$. If thermal energy Q flows from system A at temperature T_A to system B at temperature T_B (and without any mechanical work done on or by either system), then $\Delta S_A = -Q/(k_B T_A)$, and $\Delta S_B = +Q/(k_B T_B)$.

If N molecules of ideal gas go from an equilibrium state with temperature T_i and volume V_i to a new equilibrium state with temperature T_f and volume V_f, the change in entropy of the gas is (where C_V is heat capacity per particle at constant volume)

$$\Delta S_{\text{gas}} = S_f - S_i = \frac{N C_V}{k_B} \ln(T_f/T_i) + N \ln(V_f/V_i)$$

The efficiency of a heat engine is

$$\eta \ = \ \frac{W_{\text{out}} - W_{\text{in}}}{Q_{\text{in}}} \ = \ \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} \ \leq \ \frac{T_{\text{in}} - T_{\text{out}}}{T_{\text{in}}}$$

In the special case of an ideal (or Carnot, or "reversible") heat engine, the "$\leq$" becomes "=" (which you can prove by using $\Delta S_{\text{env}} = \frac{Q_{\text{out}}}{k_B T_{\text{out}}} - \frac{Q_{\text{in}}}{k_B T_{\text{in}}} = 0$).

A heat pump moves thermal energy from a "low" temperature T_L to a "high" temperature T_H. The coefficient of performance (COP) for heating is (note that "out" means the output of the heat pump, not the outdoors)

$$\text{COP}_{\text{heating}} \ = \ \frac{Q_{\text{out}}}{W_{\text{in}} - W_{\text{out}}} \ = \ \frac{Q_{\text{out}}}{Q_{\text{out}} - Q_{\text{in}}} \ \leq \ \frac{T_{\text{out}}}{T_{\text{out}} - T_{\text{in}}} \ = \ \frac{T_H}{T_H - T_L}$$

where the "$\leq$" is "=" for an ideal heat pump. For COP for cooling is

$$\text{COP}_{\text{cooling}} \ = \ \frac{Q_{\text{in}}}{W_{\text{in}} - W_{\text{out}}} \ = \ \frac{Q_{\text{in}}}{Q_{\text{out}} - Q_{\text{in}}} \ \leq \ \frac{T_{\text{in}}}{T_{\text{out}} - T_{\text{in}}} \ = \ \frac{T_L}{T_H - T_L}$$

where once again "$\leq$" becomes "$=$" if $\Delta S_{\rm env} = 0$ for a complete cycle.

Electricity

The electric charge on a proton is $+e = 1.602 \times 10^{-19}$ C (C = coulomb). The electric charge on an electron is $-e$.

Coulomb's law: charged particle B exerts electrostatic force ON charged particle A:

$$\vec{F}^{\rm elec}_{\rm B\ on\ A} \;=\; -\frac{1}{4\pi\epsilon_0}\,\frac{q_A q_B}{r^2_{AB}}\,\hat{r}_{A\to B} \;=\; \frac{q_A q_B}{4\pi\epsilon_0}\,\frac{\vec{r}_A - \vec{r}_B}{|\vec{r}_A - \vec{r}_B|^3}$$

where I used $\hat{r}_{A\to B} = -\frac{\vec{r}_A - \vec{r}_B}{|\vec{r}_A - \vec{r}_B|}$. The constant is $\epsilon_0 = 8.854 \times 10^{-12}\ \frac{\rm C^2}{\rm N \cdot m^2}$, or equivalently

$$k = \frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9\ \frac{\rm N \cdot m^2}{\rm C^2}$$

The force is repulsive (force on A points away from B) if the two charges have the same sign, and is attractive (force on A points toward B) if the two charges have opposite signs.

The electrostatic force exerted by N other particles ON particle A is

$$\vec{F}_{\rm on\ A} = \sum_{i=1}^{N} \frac{q_A q_i}{4\pi\epsilon_0}\,\frac{\vec{r}_A - \vec{r}_i}{|\vec{r}_A - \vec{r}_i|^3}$$

Writing out the components of the force (on A) for clarity:

$$F_x = \sum_i \frac{q_A q_i}{4\pi\epsilon_0}\,\frac{x_A - x_i}{((x_A - x_i)^2 + (y_A - y_i)^2 + (z_A - z_i)^2)^{3/2}}$$

$$F_y = \sum_i \frac{q_A q_i}{4\pi\epsilon_0}\,\frac{y_A - y_i}{((x_A - x_i)^2 + (y_A - y_i)^2 + (z_A - z_i)^2)^{3/2}}$$

$$F_z = \sum_i \frac{q_A q_i}{4\pi\epsilon_0}\,\frac{z_A - z_i}{((x_A - x_i)^2 + (y_A - y_i)^2 + (z_A - z_i)^2)^{3/2}}$$

The electric field $\vec{E}$ at a point $\vec{r} = (x, y, z)$ is the electrostatic force-per-unit-charge ($\vec{E} = \vec{F}/q$) that a test charge q would experience if placed at position $\vec{r}$. The electric field created by N particles is

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0}\,\frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

or writing out the components,

$$E_x(x,y,z) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \, \frac{x - x_i}{((x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2)^{3/2}}$$

$$E_y(x,y,z) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \, \frac{y - y_i}{((x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2)^{3/2}}$$

$$E_z(x,y,z) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \, \frac{z - z_i}{((x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2)^{3/2}}$$

The electric field $\vec{E}(\vec{r})$ due to a charge q placed at the origin $(0,0,0)$ is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 \; r^2} \; \hat{r}$$

where $\hat{r}$ is the unit vector pointing radially away from the origin.

Note that for a point that is outside of a uniform spherical shell of charge ("shell" = the thin surface of a sphere), $\vec{E}$ is the same as if all of the charge were at the center of the sphere. A uniform spherical shell of radius R contributes nothing (zero) to the electric field for $r < R$. (You may remember an analogous result for gravity: `http://en.wikipedia.org/wiki/Shell_theorem`) Using Gauss's law is the easiest way to show that this is true.

Gauss's law states that the electric flux (the "flow" of $\vec{E}$ field lines) through the closed surface of an arbitrary volume is

$$\Phi_E \equiv \int_{\text{surface}} E_\perp \; \mathrm{d}A = Q_{\text{enclosed}}/\epsilon_0$$

To calculate Φ_E, you sum up the area of the enclosing surface, weighting each area by the normal component (i.e. $\perp$ to the surface) of $\vec{E}$. Gauss's law is most useful if you can choose each face of your enclosing surface so that either (a) $\vec{E}$ is $\parallel$ to the face (in which case the flux through that face is zero), or (b) $\vec{E}$ is constant and $\perp$ to the face, in which case you just multiply the area of the face by E (or $-E$) for outward (or inward) pointing $\vec{E}$.

Below are one figure and one "procedure box" from Eric Mazur's chapter 24, describing how to use Gauss's law.

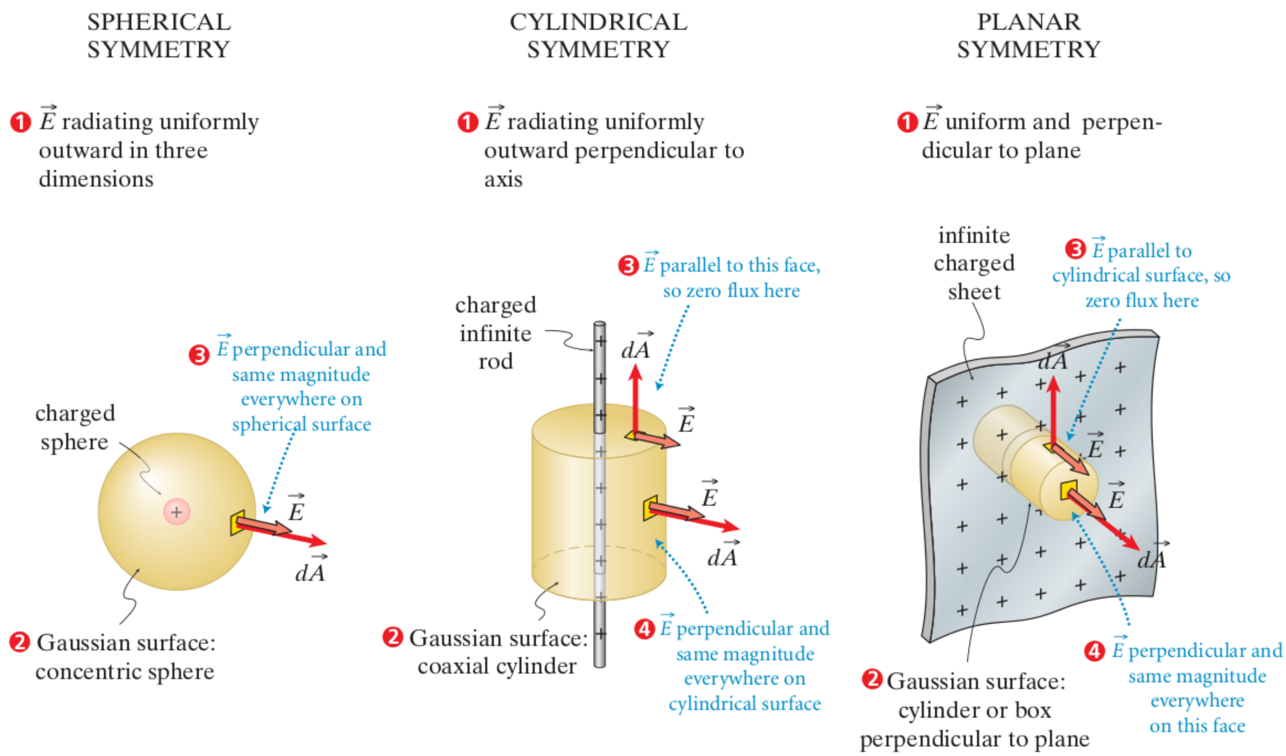


Figure 24.27 Applying Gauss's law to determine the electric fields of charge distributions exhibiting spherical, cylindrical, or planar symmetry.

PROCEDURE: Calculating the electric field using Gauss's Law

Gauss's Law allows you to calculate the electric field for charge distributions that exhibit spherical, cylindrical, or planar symmetry without having to carry out any integrations.

1. Identify the symmetry of the charge distribution. This symmetry determines the general pattern of the electric field and the type of Gaussian surface you should use (Figure 24.27)

2. Sketch the charge distribution and the electric field by drawing a number of field lines, remembering that the field lines start on positively charged objects and end on negatively charged ones. A two-dimensional drawing should suffice.

3. Draw a Gaussian surface such that the electric field is either parallel or perpendicular (and constant) to each face of the surface. If the charge distribution divides space into distinct regions, draw a Gaussian surface in each region where you wish to calculate the electric field.

4. For each Gaussian surface determine the charge q_{enc} enclosed by the surface.

5. For each Gaussian surface calculate the electric flux Φ_E through the surface. Express the electric flux in terms of the unknown electric field E.

6. Use Gauss's Law (Eq. 24.8) to relate q_{enc} and Φ_E and solve for E.

You can use the same general approach to determine the charge carried by a charge distribution given the electric field of a charge distribution exhibiting one of the three symmetries in Figure 24.27. Follow the same procedure, but in Steps 4–6, express q_{enc} in terms of the unknown charge q and solve for q.

The electric field lines from a point charge spread out in three dimensions. You can use Gauss's law to show that at a distance r from a single point charge q, the electric field has magnitude $E(r) = \frac{q/\epsilon_0}{4\pi r^2}$, which is the result you knew already.

The electric field lines from an infinitely long line of charge spread out in only two dimensions. You can use Gauss's law to show that at a distance r from a line charge whose charge-per-unit-length is q/L, the electric field has magnitude $E(r) = \frac{q/\epsilon_0}{2\pi rL}$, i.e. the field falls off only as $1/r$.

The electric field lines from an infinitely wide plane of charge do not spread out at all. (Since there is only one dimension available for getting away from the + charge, they must remain parallel.) You can use Gauss's law to show that at a distance r from a plane charge whose charge-per-unit-area is q/A, the electric field has magnitude $E(r) = \frac{q/\epsilon_0}{2A}$, i.e. the field has constant magnitude $\frac{q}{2A\epsilon_0}$. (If the plane of charge sits on only one surface of a conductor, as in a parallel-plate capacitor, then there is no factor of two, because the field is nonzero only between the two plates: then $E = \frac{q}{\epsilon_0 A}$.)

Electric potential (a.k.a. voltage) is analogous to the elevation on a topo map. Electric potential is *potential energy per unit charge* for a small test charge q

$$V = U/q$$

The work that you need to do to move a charge q across a potential difference ΔV (like pushing a ball up a hill, but mind the sign of q) is

$$W = q\Delta V$$

The electric field always points in the "downhill" direction: the direction in which V decreases most rapidly.

$$E_x = -\frac{\mathrm{d}V}{\mathrm{d}x}, \quad E_y = -\frac{\mathrm{d}V}{\mathrm{d}y}, \quad E_z = -\frac{\mathrm{d}V}{\mathrm{d}z}$$

So I can measure the strength of an electric field equivalently as $\frac{\text{newtons}}{\text{coulomb}}$ or $\frac{\text{volts}}{\text{meter}}$. (These two units are equal.) The electric field $\vec{E}$ measures

- force per unit charge
- how rapidly potential varies with position
- direction in which potential decreases most rapidly

If the point (x, y, z) is a distance d_i away from each of N point charges q_i, then (taking $V = 0$ at $r = \infty$, and integrating $-E(r)$ from $r = \infty$ to $r = d_i$)

$$V(x, y, z) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0 d_i}$$

where $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$.

Energy units useful for connecting physics with chemistry: since $e = 1.602 \times 10^{-19}$ C, an electron volt is 1 eV $= 1.602 \times 10^{-19}$ J. And since a mole is 6.022×10^{23} of something, 1 kJ/mol $= (1000 \text{ J})/(6.022 \times 10^{23}) = 1.66 \times 10^{-21}$ J.

When you accumulate electric charge Q on a single conductor (taking $V = 0$ at $r = \infty$), the potential V of the conductor is proportional to the accumulated charge Q. When you accumulate charge $+Q$ on one conductor and charge $-Q$ on a nearby conductor, the potential difference ΔV between the two conductors is proportional to Q. In either case, the constant of proportionality is a geometrical factor called the **capacitance**:

$$Q = CV$$

Usually you figure out C by drawing a picture of the conductor(s) with charge Q in place, then using Gauss's law to figure out $\vec{E}$, then using $E_x = -\mathrm{d}V/\mathrm{d}x$, etc., and integrating to find V. Then you divide to get $C = Q/V$.

For a single conducting **sphere of radius** R, you find $\mathbf{C = 4\pi\epsilon_0 R}$. For two **parallel plates** of area A separated by distance d, you find $\mathbf{C = \epsilon_0 A/d}$. If you understand Gauss's law, it can be fun to derive that for two long coaxial cylinders of length L and radii r_1 and r_2, the capacitance is $C = 2\pi\epsilon_0 L/\ln(r_2/r_1)$.

The potential energy stored in a capacitor is $U = \frac{1}{2}CV^2 = \frac{1}{2}Q^2/C$

If you stick an electrical insulator (called a *dielectric*) between the plates of a capacitor, the factor ϵ_0 in the capacitance is replaced by $\epsilon = K\epsilon_0$, where K is known as the **dielectric constant**. The way this comes about is that the + and − charges inside the dielectric material separate just a tiny bit, responding in proportion to the external electric field. They are only able to move maybe a few angstroms in response to the external field, so their movement only partially cancels out $\vec{E}$. This partial cancellation replaces $\vec{E}$ inside the dielectric with $\vec{E}/K$. So then when you compute ΔV for a given Q, you get a number that is smaller by a factor of K. Smaller ΔV for a given K means that $C \to KC$.

Ohm's law for a resistor: $\Delta V = IR$, where ΔV is the voltage drop across the resistor (the potential difference between the two terminals of the resistor), and I is the current through the resistor.

For a resistor of length L and constant cross-sectional area A, made from material of **conductivity** σ, the resistance is $R = L/(\sigma A)$. Or in terms of **resistivity** $\rho = 1/\sigma$, $R = \rho L/A$.

Suppose that a direct current I flows through a circuit element, from terminal a to terminal b, and that the volage *drop* across that circuit element is $\Delta V = V_b - V_a > 0$. Then the **power dissipated** by (or perhaps stored in or otherwise consumed by) that circuit element is $I\Delta V$. (In the case of alternating current, you can often just replace I and ΔV by their rms values and use $I_{\text{rms}}\Delta V_{\text{rms}}$ for power, but in general you need to account for the possibility that I and ΔV are out of phase with one another by some angle ϕ, in which case the power is $I_{\text{rms}}\Delta V_{\text{rms}}\cos(\phi)$. In the unlikely event that you want the details behind this, see Mazur's chapter 32—which we won't have time to cover, but you're welcome to read it for extra credit.)

Combining this last result with Ohm's law, the **power dissipated in a resistor** is $P = I^2R = V^2/R$.

Junction rule for circuits in steady state (charge conservation): $\sum I_{\text{in}} = \sum I_{\text{out}}$. This is like saying that per unit time, the number of cars entering an intersection equals the number of cars leaving the intersection—which is true for a steady flow of traffic.

Loop rule for circuits (energy conservation): $\sum \Delta V = 0$. As you go around the loop, add up the voltage *gains*: $\Delta V = -IR$ for each resistor, and $\Delta V = +\mathcal{E}$ for each battery, if your loop exits the battery at the (+) terminal and re-enters the battery at the (−) terminal. (If your loop goes through the battery in the opposite direction, then $\Delta V = -\mathcal{E}$.) This is like saying that if I go backpacking for a few days in Yosemite, taking a "loop" route that begins and ends at the same trail head, the sum of my uphill ascents equals the sum of my downhill descents, because I end at the same elevation at which I began.

A consequence of the above two rules (you can draw series and parallel circuits, find the current drawn from a battery of given $\mathcal{E}$, and find $R_{\text{combined}} = \mathcal{E}/I_{\text{battery}}$) is that the combined resistance R for two resistors R_1 and R_2 **in series** is $R = R_1 + R_2$. Another consequence is that **in parallel**, $1/R = 1/R_1 + 1/R_2$.

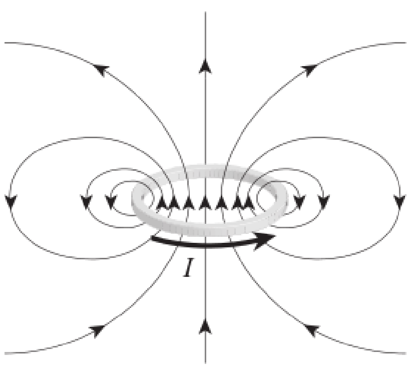
Using the results from the previous paragraph, you can show that putting n copies of the same resistor R_1 in series gives you $R = nR_1$. Putting n copies of the same resistor R_1 in parallel gives you $R = R_1/n$.

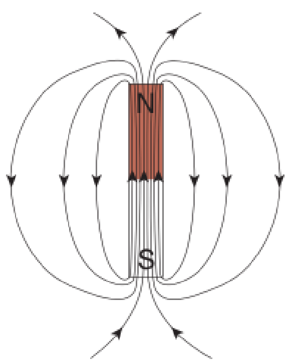
Magnetism

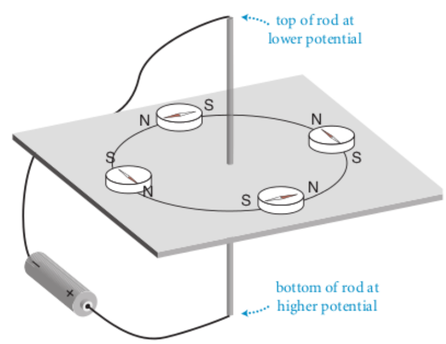
Opposite poles of magnets attract; like poles repel. So N attracts S; N repels N.

Magnetic field lines flow out of the north pole of a magnet and back into the south pole. (Inside the magnet, the field lines continue from S to N, completing the loop. Because magnetic charges ("magnetic monopoles") don't exist, magnetic field lines always form closed loops.) So the magnetic field outside of a bar magnet points away from the north pole and toward the south pole.

The arrow of a compass is the north pole of a magnet. It actually points toward the **south** magnetic pole of the earth, which is located near the geographic north pole, confusingly enough. (Because opposite poles attract, there's no way to avoid this, alas.) If you put a compass into a magnetic field, the arrow (with the "N" marking) will line up with the $\vec{B}$ field direction (or for a flat compass, the projection of the $\vec{B}$ field direction into the plane of the compass).







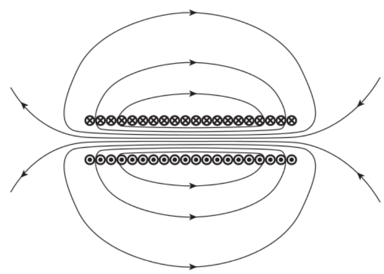
$\vec{B}$ field lines encircle an electric current (or a charged particle in motion). If you orient your right thumb in the direction in which positive current flows (or positively charged particles move), the fingers of your right hand will curl around in the direction of $\vec{B}$.

The right-hand rule is usually all you need to figure out the direction of $\vec{B}$. For those rare cases (not in this course) in which you need to figure out the magnitude, use Ampere's law: going around a closed loop, sum up the distance traveled times the component of $\vec{B}$ in the direction of travel. This sum (or integral) equals the constant $\mu_0 = 4\pi \times 10^{-7}$ T/(A $\cdot$ m) times the net current enclosed by your Amperian loop.

$$\oint_{\substack{\text{closed}\\ \text{loop}}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \; I_{\text{enclosed}}$$

Using Ampere's law, you can derive the magnitude of $\vec{B}$ at distance r around a long, straight wire carrying current I to be $B = (\mu_0 I)/(2\pi r)$.

More importantly, Ampere's law also tells you that the magnitude of $\vec{B}$ **inside** a long solenoid (a cylinder with wire wrapped around it, the most common shape for an electromagnet) is $B = (N/L)\mu_0 I$, where N/L is the number of turns of wire per unit length and I is the current flowing in the wire.



The force on a moving charged particle due to magnetic field $\vec{B}$ is $\vec{F} = q\vec{v} \times \vec{B}$, where q is the particle's electric charge and v is the particle's velocity. The magnitude of $\vec{F}$ is $|\vec{F}| = qvB \sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{B}$. The force is largest when $\vec{v}$ and $\vec{B}$ are perpendicular and is zero when $\vec{v}$ and $\vec{B}$ are parallel or antiparallel. You can figure out the direction of $\vec{F}$ for positive q by first pointing the fingers of your right hand in the $\vec{v}$ direction, then curling them in the $\vec{B}$ direction. Your right thumb will then point in the $\vec{F}$ direction. (For negative q, your right thumb winds up pointing in the $-\vec{F}$ direction.) The magnetic force is always perpendicular both to $\vec{v}$ and to $\vec{B}$.

An equivalent version of the above force law is more useful for calculating the force on a current-carrying wire in a magnetic field: $\vec{F} = I\vec{\ell} \times \vec{B}$. You use the same right-hand rule as in the previous paragraph, with the direction of positive current instead of the velocity. If the current flow is perpendicular to $\vec{B}$, then the magnitude of the force is $F = I\ell B$. Or if the current flow makes an angle θ with $\vec{B}$, then $F = I\ell B \sin\theta$.

The *magnetic flux* through a surface is of area A is $\Phi_B = BA\cos\theta$, where θ is the angle between the $\vec{B}$ field lines and the surface normal. (The "surface normal," which you may encounter in computer graphics or CAD software, is a vector that is locally perpendicular to the surface; for example, the surface normal of a spherical surface always points radially.) So the flux is maximum when $\vec{B}$ is normal (perpendicular) to the surface, and is zero when $\vec{B}$ is parallel to the surface. If the surface is not flat or $\vec{B}$ is not constant, you divide the surface up into many small pieces and add them. So more formally $\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$, where $\vec{A}$ points along the surface normal. Intuitively, you can think of Φ_B as proportional to the net number of $\vec{B}$ field lines that pierce the surface in the "outward" direction (you have to call one side of the surface the outside and one the inside, and count up outgoing minus ingoing field lines).

A closed loop of wire defines a surface. (Think of the surface of the soap bubble that would form with the loop as a frame.) When the magnetic flux through that surface changes, an emf (i.e., a voltage) is induced around the loop: $\mathcal{E} = -\mathrm{d}\Phi_B/\mathrm{d}t$. You can change Φ_B by changing the magnitude of $\vec{B}$, by changing the direction of $\vec{B}$, or by changing the orientation of the loop. The minus sign reminds us that the induced current flows in the direction that creates a $\vec{B}$ field that opposes the change in magnetic flux. So if the loop is in the xy plane, and $\vec{B}$ points along the z axis and is increasing, then $\mathcal{E}$ induces a clockwise current (as seen from $+z$), because a clockwise current creates a magnetic field pointing along in the $-z$ direction, which is opposite $\mathrm{d}\vec{B}/\mathrm{d}t$. (See "Faraday's law" and "Lenz's law" in the book.)

If the wire is coiled around so that the $\vec{B}$ field lines pass through the coil N times, then Φ_B is N times as large.

In a transformer, the primary coil is connected to a source of AC voltage V_p, and the secondary coil is connected to a load to which you want to supply voltage V_s. Then $V_s/V_p = N_s/N_p$, where N_s is the number of times the secondary coil is wound around the iron core, and N_p is the number of times the primary coil is would around the iron core. The purpose of the iron is to ensure that all of the magnetic field lines passing through one coil also pass through the other coil. $V_s > V_p$ is called "step up," and $V_s < V_p$ is called a "step down" transformer. For an ideal transformer (most real transformers nowadays are not far from ideal), 100% of the power supplied by the primary circuit is delivered to the secondary circuit, i.e. $V_p I_p = V_s I_s$.

What I hope you remember about electricity and magnetism from this course is (a) mainly how circuits work, and (b) this synopsis:

- Electric charge creates an electric field.
- Moving electric charge (i.e. electric current) creates a magnetic field.
- Force $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ on charged particles.
  - electromagnet, speaker, doorbell, mass spectrometer, electric motor
- Changing magnetic flux induces "emf" (voltage) $\mathcal{E}$ in loop
  - electric generator, AC transformer
- Equivalently, changing magnetic field creates an electric field.
- Changing electric field creates a magnetic field.

- Last two together allow EM waves to propagate
  - wireless telegraph, radio, cell phone, and even light

Quantum mechanics / atoms

Light is emitted and absorbed in discrete "quanta" called photons. The energy E_γ of a photon is given by

$$E_\gamma = hf = \frac{hc}{\lambda}$$

where $h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s}$ is Planck's constant, $c = 2.9979 \times 10^{8}\ \mathrm{m/s}$ is the speed of light, f is frequency (in cycles/second, or Hz), and λ is wavelength (in meters).

The spectrum of thermal (black-body) radiation is very broad, but the *peak* (most probable) wavelength is inversely proportional to temperature: hotter objects tend to emit higher-energy (thus lower wavelength) photons:

$$\lambda_{\mathrm{peak}} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{T}$$

with λ_{peak} in meters and T in kelvins. See `http://en.wikipedia.org/wiki/Wien%27s_displacement_law` . You can see qualitatively where this comes from by assuming that the photon energy corresponding to λ_{peak} is $E_\gamma = \alpha k_B T$, where k_B is Boltzmann's constant from thermal physics, and α is some numerical constant. It turns out that $\alpha = 4.96$ gives you the experimentally correct result: $\lambda_{\mathrm{peak}} = (hc)/(4.96 k_B T)$.