Physics 8 — Monday, December 2, 2019

- Final exam (25%) is Thu, Dec 12, noon–2pm, DRL A1.
- I'll try to book a room for a review session on Wed, Dec 11, preferably mid-afternoon.
- Pick up take-home practice exam (10%) in back of room.
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- 4 previous years' exams & practice exams are at `http://positron.hep.upenn.edu/p8/files/oldexams`
- Periodic motion (oscillation, vibration) is our last topic this term. Alas, this year's exam schedule doesn't allow us to include it in the homework or the exam.

Vibrations/oscillations

- Are ubiquitous (look around — or listen — for examples!) because anything in stable equilibrium can oscillate about the equilibrium point. (Illustration.)

Picture a ball at the bottom of a round container. Is it in stable equilibrium at the bottom? What happens if I slide it over a bit and then let go of it?

- The restoring force pushes it back toward the equilibrium position. Once it reaches the equilibrium position, the net force is zero, but by that point the ball is in motion, so it continues past the equilibrium point until the restoring force eventually reverses its direction. It keeps moving back and forth until eventually the energy is dissipated by friction, and the ball comes to rest in the equilibrium position.
- Contrast with *neutral* or *unstable* equilibrium: no restoring force in these cases.

Oscillations / vibrations

- The restoring force that keeps an object stable is the same restoring force that causes the object to vibrate when displaced.
- The simplest form for a restoring force is Hooke's law:

$$F_x = -k\,(x - x_0)$$

- A linear restoring force is the most common case, for small displacements. We study it because it is ubiquitous and because it is relatively easy to analyze.
- If there is a linear restoring force (i.e. if the force is proportional to the displacement) and negligible friction, then the math works out cleanly with sines and cosines, and we call the motion *Simple Harmonic Motion*.

skip math — here in case you're curious

Hooke's law for a block on a horizontal spring is

$$F_x = -k\ (x - x_0)$$

(Note: for vertical orientation, the equilibrium position is offset downward by mg/k, after which the math is identical.)

Newton's 2nd law for the block of mass m then reads

$$ma_x = -k\ (x - x_0)$$

To simplify the math, let $x_0 = 0$ for the moment. Then

$$ma_x = -kx$$

$$m\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx$$

A function whose second derivative is proportional to the original function (with a negative coefficient) is a sine or a cosine.

$$x(t) = A\ \sin(\omega t + \phi_i)$$

Plugging **[skip math — it's here for your curiosity]**

$$x(t) = A\ \sin(\omega t + \phi_i)$$

into

$$ma_x = -kx$$

works, using "angular frequency" ω (radians/second)

$$\omega = \sqrt{\frac{k}{m}}$$

Or (more familiar) "frequency" (cycles/second, or Hz)

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

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check:

$$v_x(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \omega A\ \cos(\omega t + \phi_i)$$

$$ma_x(t) = m\ \frac{\mathrm{d}v_x}{\mathrm{d}t} = -m\omega^2 A\ \sin(\omega t + \phi_i) = -kA\ \sin(\omega t + \phi_i) = -kx$$

For a mass oscillating on a spring at its "natural frequency," i.e. the frequency at which it oscillates if I pluck it or whack it and then leave it alone

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}}$$

and the motion is sinusoidal in time:

$$x(t) = x_{\mathrm{eq}} + A\ \sin(\omega_0 t + \phi_i)$$

- x_{eq} is equilibrium position (usually we choose $x_{\mathrm{eq}} = 0$)
- A is called the amplitude
- The "initial phase" ϕ_i tells you what's happening at $t = 0$
- $\phi_i = 0$ or π means displacement w.r.t. x_{eq} is zero at $t = 0$
- $\phi_i = \pm\pi/2$ means displacement is max(min)imum at $t = 0$
- notice $\sin(\omega t \pm \pi/2) = \pm\cos(\omega t)$

$$x(t) = x_{\text{eq}} + A \sin(\omega_0 t + \phi_i)$$

Writing $x(t)$ this way is usually more complicated than necessary. The most common cases for ϕ_i are:

- $\phi_i = 0$: at $t = 0$, $(x - x_{\text{eq}}) = 0$ and $v_x > 0$ (maximum)

$$x(t) = x_{\text{eq}} + A \sin(\omega_0 t)$$

- $\phi_i = \pi/2$: at $t = 0$, $(x - x_{\text{eq}}) > 0$ (maximum) and $v_x = 0$

$$x(t) = x_{\text{eq}} + A \cos(\omega_0 t)$$

- $\phi_i = \pi$: at $t = 0$, $(x - x_{\text{eq}}) = 0$ and $v_x < 0$ (minimum)

$$x(t) = x_{\text{eq}} - A \sin(\omega_0 t)$$

- $\phi_i = -\pi/2$: at $t = 0$, $(x - x_{\text{eq}}) < 0$ (minimum) and $v_x = 0$

$$x(t) = x_{\text{eq}} - A \cos(\omega_0 t)$$

As another simplification, usually we define the x axis so that $x_{\mathrm{eq}} = 0$. Then for the two most common cases:

- at $t = 0, \quad x = 0$ and $v_x > 0$

$$x(t) = A \; \sin(\omega_0 t)$$

$$v_x(t) = \omega_0 A \; \cos(\omega_0 t)$$

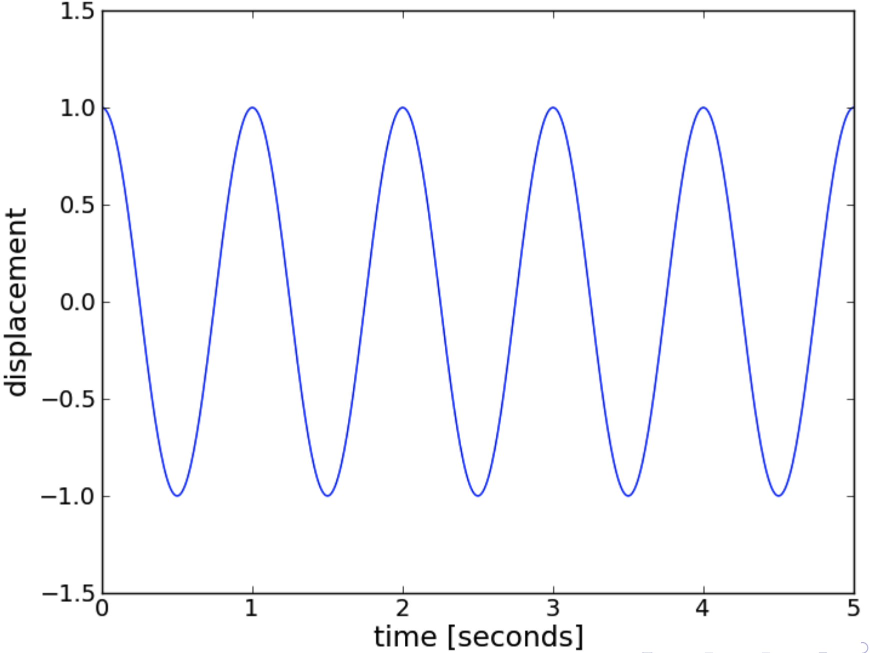
- at $t = 0, \quad x > 0$ and $v_x = 0$

$$x(t) = A \; \cos(\omega_0 t)$$

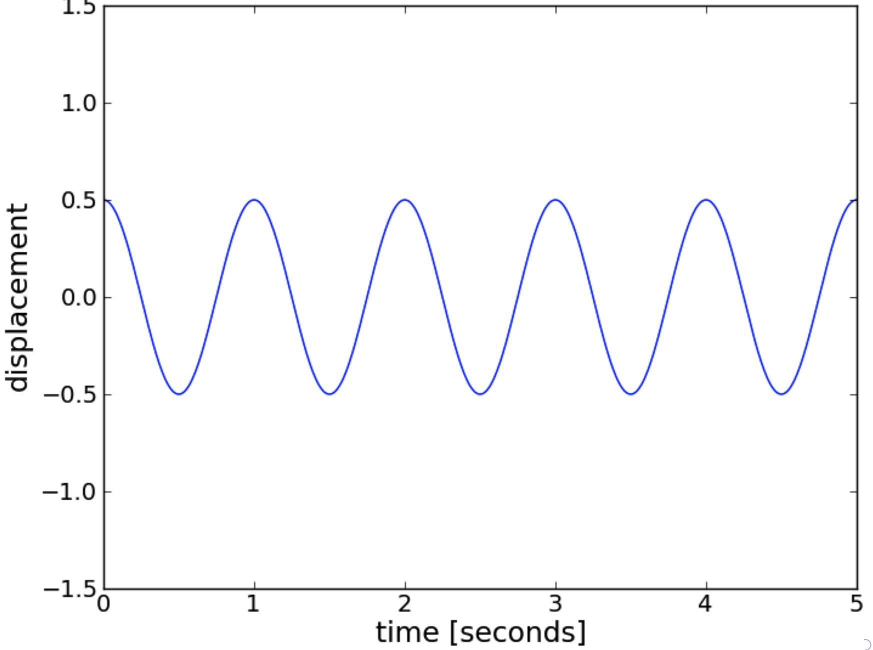
$$v_x(t) = -\omega_0 A \; \sin(\omega_0 t)$$

Let's try this with graphs instead of equations. The next few graphs will assume that we choose $x_{\mathrm{eq}} = 0$.

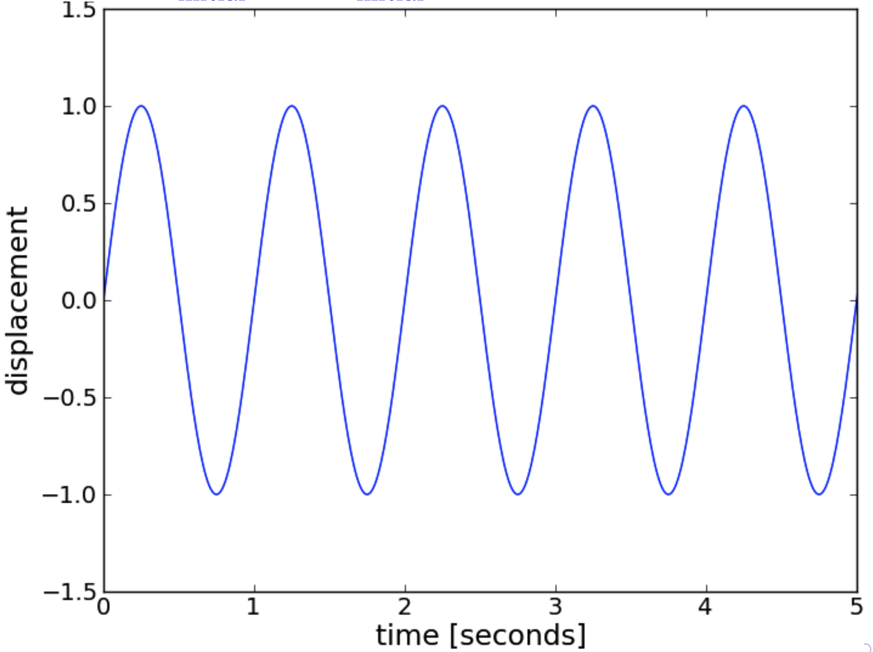
Oscillation: $v_{\text{initial}} = 0$, $x_{\text{initial}} > 0$: looks like a **cosine**



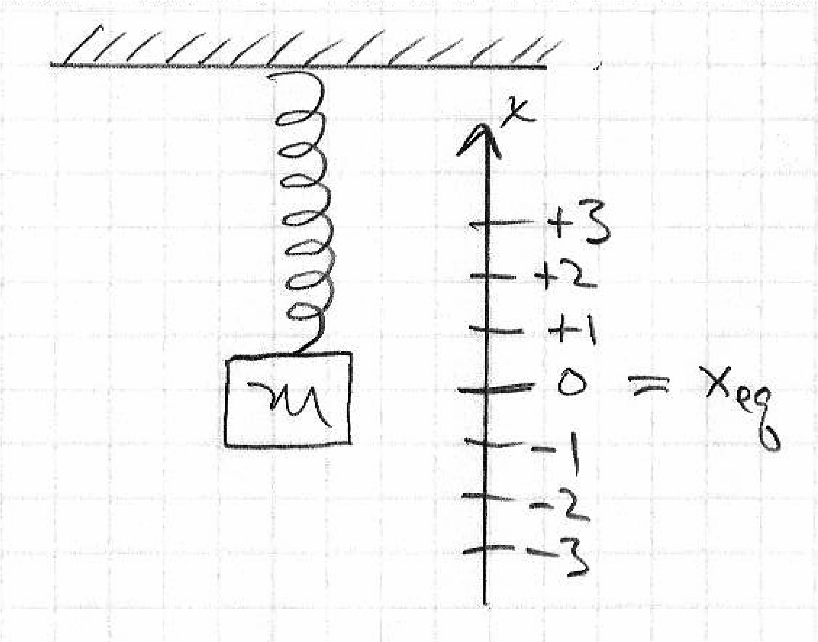
now reduce amplitude by half from previous slide

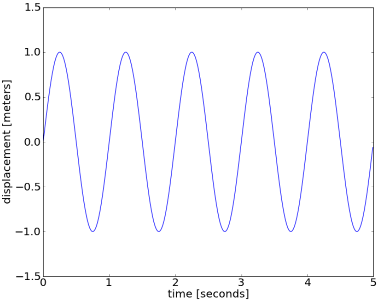


Oscillation: $v_{\text{initial}} > 0$, $x_{\text{initial}} = 0$: looks like a **sine**

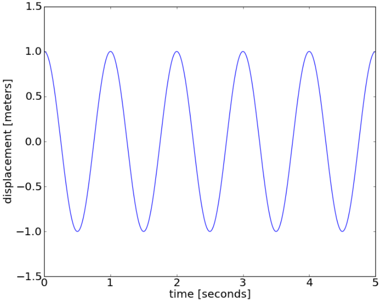


Let's try some examples using a coordinate system that looks like this. So $x_{eq} = 0$ and the x axis points upward.




displacement [meters]
time [seconds]
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
1
2
3
4
5

A


displacement [meters]
time [seconds]
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0.5
0.0
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−1.0
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0
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3
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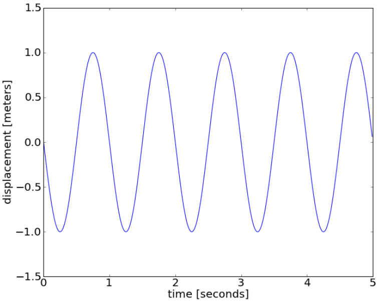
5

B


displacement [meters]
time [seconds]
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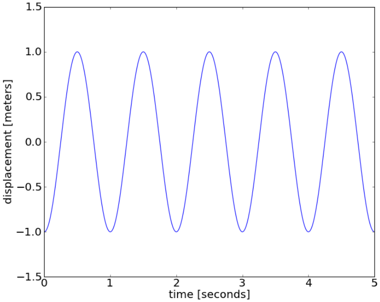
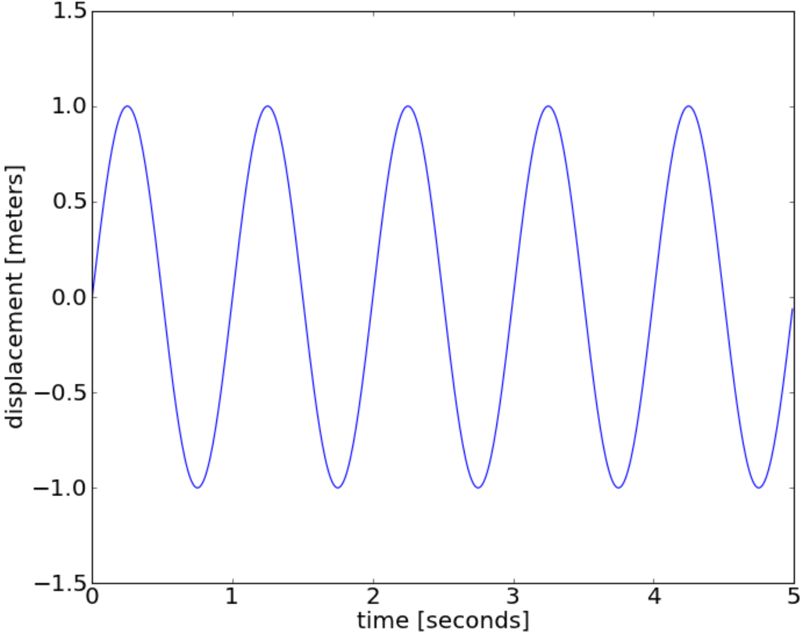
0.0
−0.5
−1.0
−1.5
0
1
2
3
4
5

C

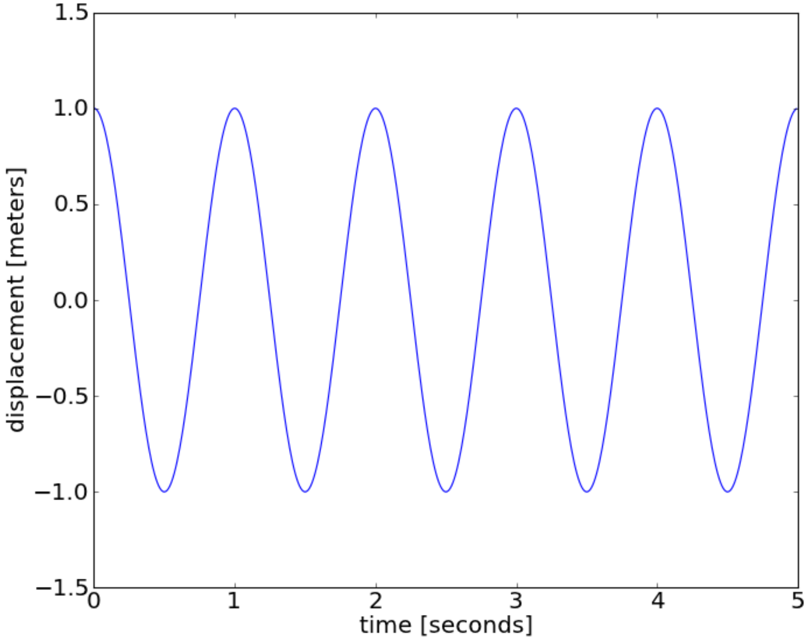

displacement [meters]
time [seconds]
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
1
2
3
4
5

D

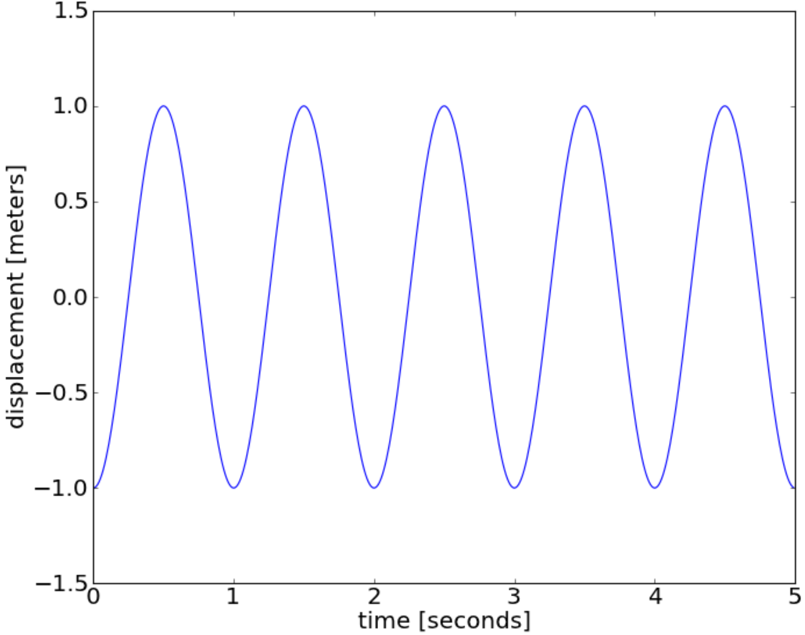
What's the amplitude of this motion? Period? Frequency?



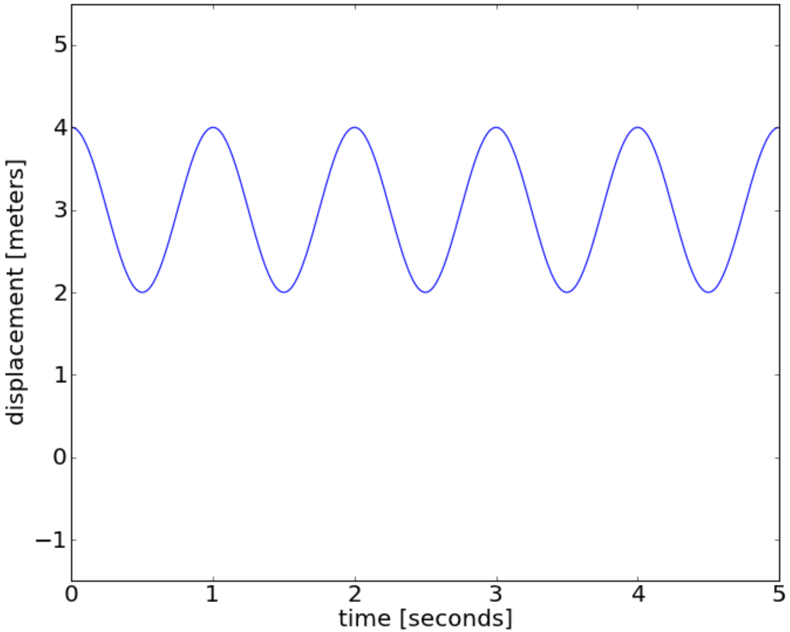
What's the amplitude of this motion?



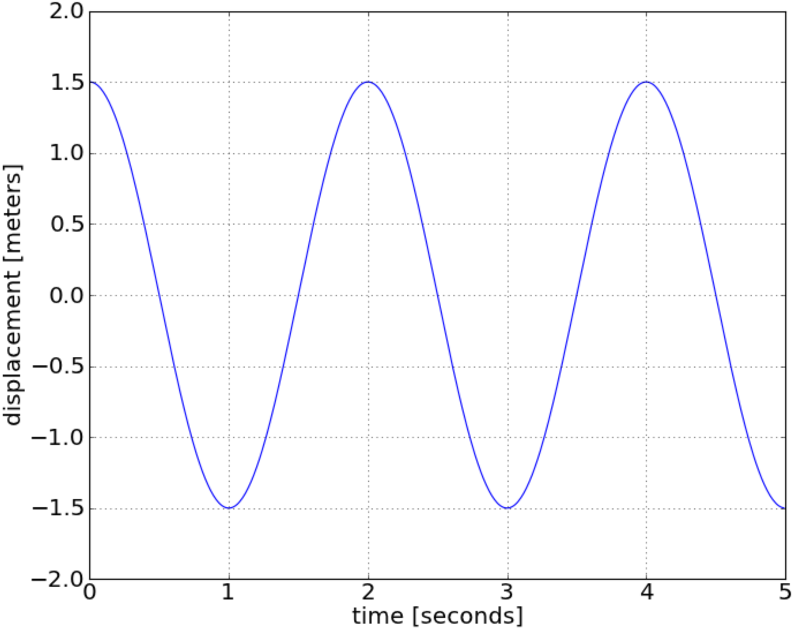
What's the amplitude of this motion?



What's the amplitude of this motion? What is x_{eq}?



Amplitude? x_{eq}? Period? Frequency? Angular frequency?

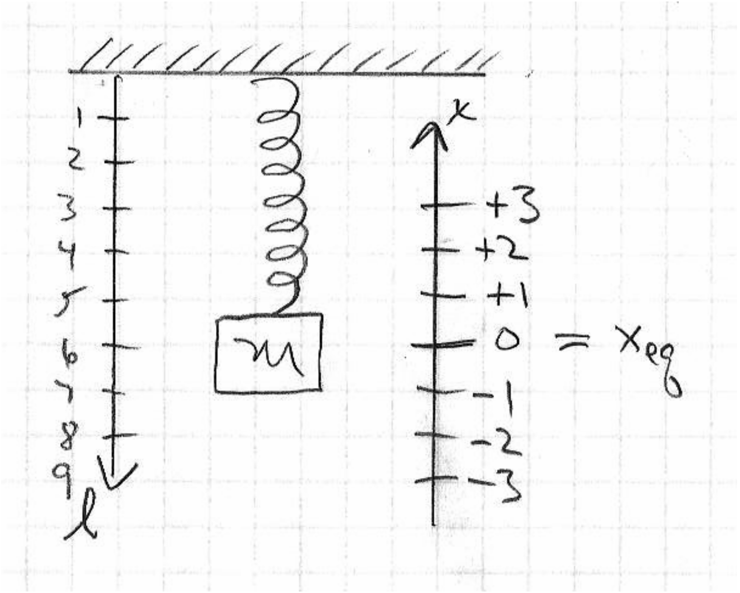


- Worth remembering: natural frequency for a mass on a spring

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

- double k $\rightarrow$ multiply f by $\sqrt{2}$
- double m $\rightarrow$ divide f by $\sqrt{2}$

- For a wide range of equilibrium situations in which the restoring force is provided by some form of elasticity,
  - more stiffness $\rightarrow$ higher f
  - more mass $\rightarrow$ lower f
- See same $\sqrt{\frac{\text{stiffness}}{\text{inertia}}}$ trend in beams, skyscrapers, etc.
- But pendulum is an exception, because restoring force $\propto$ m. We'll see in a moment.
- Another surprising result: frequency of oscillation is independent of amplitude
- Let's use a much stiffer spring and a much larger mass to illustrate this last result! How can we measure k?



Caution: for this situation, if you want to graph the length of the spring vs. time, the "length" coordinate increases in the **downward** direction, and "$\ell = 0$" is at the ceiling.

We wrote $x(t)$ in terms of ω = "natural angular frequency:"

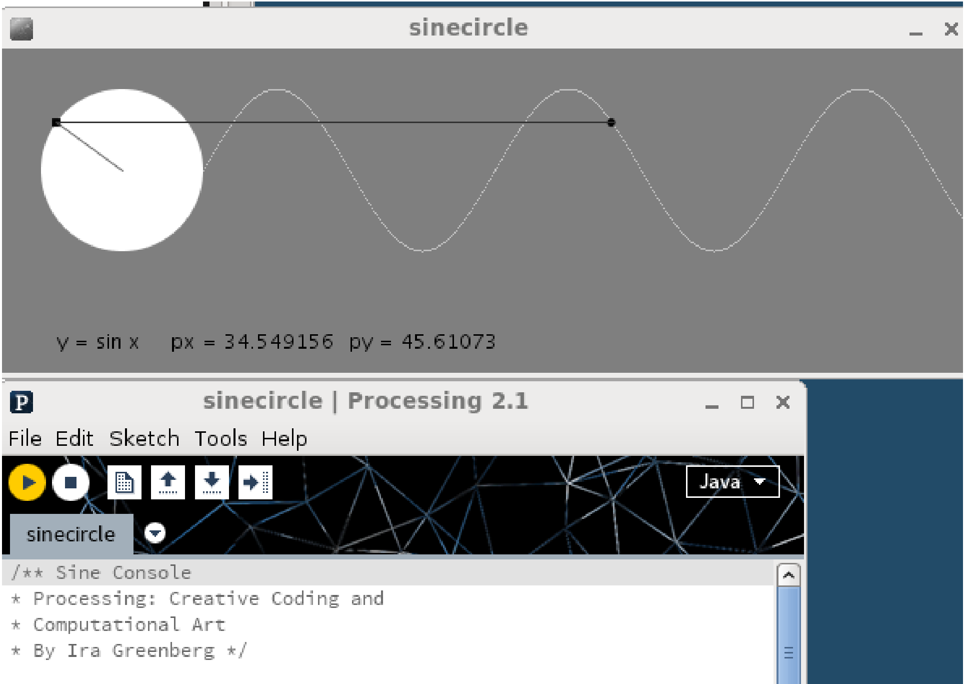
$$x = A\sin(\omega t + \phi_i)$$

but we could have equivalently used f = "natural frequency:"

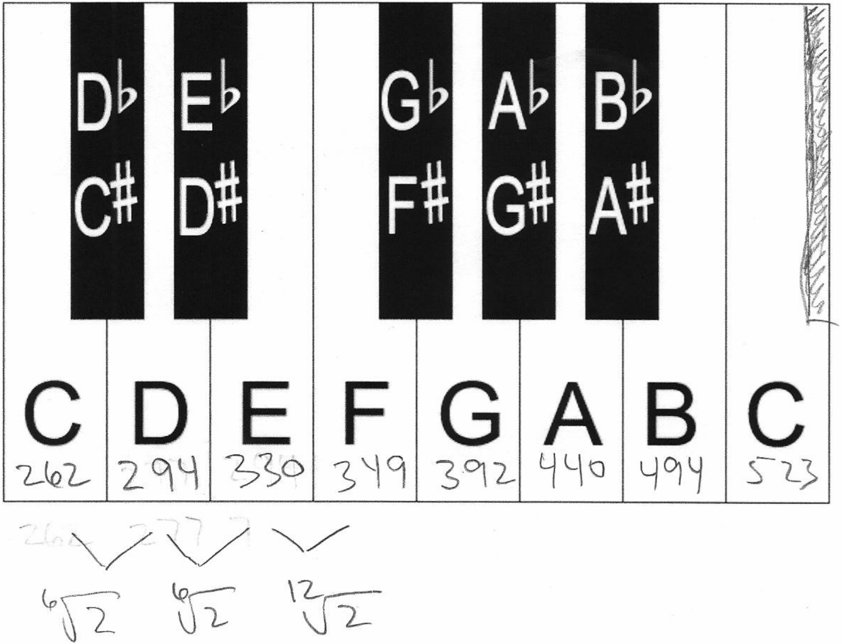
$$x = A\sin(2\pi f\ t + \phi_i)$$

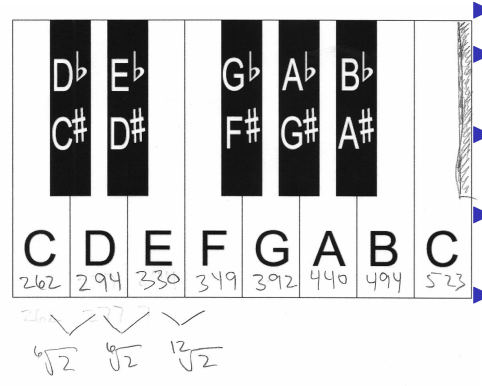
- f = *frequency*, measured in cycles/sec, or Hz (hertz)
- $\omega = \frac{f}{2\pi}$ is *angular frequency*, measured in radians/sec, or s^{-1}
- The frequency $f = 2\pi\omega$ is much more intuitive than ω
- Using ω keeps the equations cleaner — can be helpful for derivations, etc., so that you don't have to keep writing 2π

“angular velocity” ω is our old friend from studying circular motion:



"frequency" $f = \frac{\omega}{2\pi}$ is more familiar from music, etc.





- A above middle C: 440 Hz
- Middle C: 261.63 Hz
- $440 \times (\frac{1}{2})^{\frac{3}{4}} = 261.63$
- Octave = factor of 2 in frequency f
- Half step = factor of $\sqrt[12]{2}$ in frequency
- Whole step = factor of $\sqrt[6]{2}$ in frequency
- Major scale (white keys, starting from C) = (root) W W H W W W H
- Minor scale (white keys, starting from A) = (root) W H W W H W W

If the amplitude of simple harmonic motion doubles, what happens to the frequency (i.e. the natural frequency) of the system?

(A) The frequency is $1/2$ as large.

(B) The frequency is $1/\sqrt{2}$ as large.

(C) The frequency is unchanged.

(D) The frequency is $\sqrt{2}$ times as large.

(E) The frequency is 2 times as large.

If the amplitude of simple harmonic motion doubles, what happens to the energy of the system?

(A) The energy is unchanged.

(B) The energy is $\sqrt{2}$ times as large.

(C) The energy is 2 times as large.

(D) The energy is 4 times as large.

If the amplitude of simple harmonic motion doubles, what happens to the energy of the system?

(A) The energy is unchanged.

(B) The energy is $\sqrt{2}$ times as large.

(C) The energy is 2 times as large.

(D) The energy is 4 times as large.

One way to see that **(D)** is correct is to write

$$x = A\sin(\omega t) \qquad v_x = \omega A\cos(\omega t)$$

and then write out the energy

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

and see that energy is proportional to A^2.

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