phys8_slides_14.pdf

- This file has **nothing** to do with Mazur ch14, which we will not cover in this course. (The next planned Mazur chapter is ch15 in the last week of the term.)
- Onouye/Kane ch6 (cross-sectional properties), ch7 (beams I), ch8 (beams II)
- pages 17–25 of positron.hep.upenn.edu/p8/files/equations.pdf (which I'll paste into these slides for your convenience)

(Onouye/Kane ch6: cross-sectional properties)

The *centroid*, denoted $(\overline{x}, \overline{y})$, is the mass-weighted average of the centers-of-mass of the constituent parts: $\overline{x} = (\sum_i x_i m_i)/(\sum_i m_i)$, and $\overline{y} = (\sum_i y_i m_i)/(\sum_i m_i)$, where m stands for mass. To find the centroid of a continuous object, use an integral instead of a sum: $\overline{x} = (\int x\,\mathrm{d}m)/(\int \mathrm{d}m)$, and $\overline{y} = (\int y\,\mathrm{d}m)/(\int \mathrm{d}m)$. If the material is of uniform density and thickness, then you can use area A instead of mass m.

The centroid of a right triangle one side of which lies along the x axis (base b) and one side of which lies along the y axis (height h) is $(\overline{x}, \overline{y}) = (b/3, h/3)$. The area is of course $bh/2$.

If a shape has a hole in it, you can "subtract" the hole from the shape by using a negative area for the hole in the centroid calculation!

The *second moment of area* (which this book calls *moment of inertia of an area,* and most engineers and architects simply call *moment of inertia*) is most commonly given by $I_x = \int y^2 \, \mathrm{d}A$. Second moment of area is a difficult but important concept that helps to explain why an I-beam has the shape it has (with material far away from the $y = 0$ plane) and why a floor joist ("on edge") is stiff but the same board used as a plank ("on the flat") is floppy. As we'll see, a larger I_x makes a beam more stiff.

I avoid the phrase "moment of inertia" because it is ambiguous: most structures books use the phrase "moment of inertia" to refer to what I call "second moment of area," while most physics books use the phrase "moment of inertia" to refer to what Mazur calls "rotational inertia." Saying "rotational inertia" and "second moment of area" is always unambiguous, while saying "moment of inertia" is often ambiguous.

For a beam of rectangular cross-section $b \times h$ and uniform density supporting a vertical load, $I_x = bh^3/12$. Imagine a wooden beam (like a "two by ten") whose cross-section has small dimension d and large dimension D (e.g. maybe $d = 4\,\mathrm{cm}$ and $D = 20\,\mathrm{cm}$). If you orient the beam "on edge", like this ▯, then you get $I_x($▯$) = dD^3/12$. If you orient the beam "on the flat," like this ▭, then you get $I_x($▭$) = Dd^3/12$. The ratio is $I_x($▯$)/I_x($▭$) = (D/d)^2$, which is $5^2 = 25$ for the numbers given above. So the same piece of wood is $25\times$ stiffer (for these example numbers) when oriented as a joist than it is when oriented as a plank.

I_x, which represents how far the material of a beam is spread out from the $y = 0$ plane, is called I_x because if you draw a cross-section of the beam, the $y = 0$ plane is the x axis. So in cross-secton, I_x quantifies how far the material is from the x axis.

If a beam's cross-section consists of several components having cross-sectional areas A_1, A_2, A_3, vertical centroids y_1, y_2, y_3, and their own second moments of area I_{x1}, I_{x2}, I_{x3}, then you can compute the second moment of area of the composite beam using the *parallel axis theorem*:

$$I_x = I_{x1} + I_{x2} + I_{x3} + A_1 y_1^2 + A_2 y_2^2 + A_3 y_3^2 \ .$$

You could use this, for example, to find I_x for an I-beam:  . Using more general notation, the parallel axis theorem reads $I_x = \sum_i (I_{xi} + A_i y_i^2)$. Warning: the way I've written this expression, you must choose $y = 0$ to be the vertical centroid of the cross-section, i.e. you must ensure that $\overline{y} = (\sum_i y_i A_i)/(\sum_i A_i) = 0$.

The *radius of gyration* $r_x = \sqrt{I_x/A}$ is the distance from the x axis at which you could concentrate all of the beam's material (symmetrically above and below) to get the same second moment of area I_x. Notice that $I_x = Ar_x^2$. The only place you are likely to use the radius of gyration is in calculating a *slenderness ratio* of a column. (Reading O/K ch9 on columns is an extra-credit option.)

(Onouye/Kane ch7: simple beams)

The most common support configurations for beams are *simply supported* (pin beneath one end and roller beneath the other end), *overhang* (like simply supported, but ends of beam extend beyond one or both supports), and *cantilever* (one "fixed" end, and one "free" end).

A *load diagram* is basically an EFBD of the beam. Remember to include the vertical reaction forces exerted by the supports on the beam. Sometimes the load diagram is represented as a graph of the distributed load $w(x)$ (force per unit length). In load, shear, and moment diagrams, the coordinate x measures distance along the length of the beam, starting from the left end of the beam. It is confusing that this meaning of the coordinate x is different from its meaning in chapter 6 — at least y will have the same meaning here as in chapter 6.

The *shear diagram*, $V(x)$, is drawn directly below the load diagram. $V(x)$ has dimensions of force (newtons, kilonewtons, pounds, kilopounds ("kips")). If you section the beam into two halves at a distance x from the left end of the beam, the function $V(x)$ represents the upward force exerted by the left side on the right side of the beam at that section. Equivalently, $V(x) = -\int w(x)\,\mathrm{d}x$. Also equivalently, $V(x)$ is the running sum of the loads and reactions (upward minus downward) to the left of (and including) the section at x.

The *moment diagram*, $M(x)$, is drawn directly below the shear diagram. $M(x)$ has dimensions of a bending moment (or torque), i.e. force$\times$distance (newton-meters, kilonewton-meters, foot-pounds, kilopound-feet). If you section the beam into two halves at a distance x from the left end of the beam, the absolute value of the function $M(x)$ represents the magnitude of the moment (torque) that one side of the beam exerts on the other side. But the sign convention is such that "a positive moment makes the beam smile." So if the beam curves upward (smiles) under load (if $d^2Y/dx^2 > 0$) then $M > 0$, and if the beam curves downward (frowns) under load (if $d^2Y/dx^2 < 0$) then $M < 0$. Mathematically $M(x) = \int V(x)\,\mathrm{d}x$.

Since derivatives are less tricky than integrals, it may be worth remembering that $\mathrm{d}M(x)/\mathrm{d}x = V(x)$. The shear diagram $V(x)$ is the derivative (the slope) of the moment diagram $M(x)$. For distributed loads, it is also worth remembering that $\mathrm{d}V(x)/\mathrm{d}x = -w(x)$. The distributed load $w(x)$ is minus the slope of the shear diagram $V(x)$.

A free end, a pin-supported end, and a roller-supported end are all incapable of supporting a bending moment. So for any of those end conditions, $M(0) = 0$ and $M(L) = 0$. An exception is the cantilever beam, which has one free end and one fixed end. The fixed end of a cantilever has $M \neq 0$. Since a cantilever always frowns under a gravitational load, the fixed end has $M < 0$.

Sometimes one draws two additional curves beneath $M(x)$. The slope of the loaded beam, $\theta(x) = dY/dx$, is given by $EI\,\theta(x) = \int M(x)\,\mathrm{d}x$, where E is Young's modulus (elastic modulus) and I is the second moment of area (called I_x in chapter 6). If one draws $\theta(x)$, it is drawn directly below the $M(x)$ diagram. The deflected shape of the loaded beam, $Y(x)$ is given by $Y(x) = \int \theta(x)\,\mathrm{d}x$. If one draws $Y(x)$, it is drawn directly below the $\theta(x)$ diagram.

While you will probably never actually draw the $\theta(x)$ and $Y(x)$ curves, a key takeaway is that you integrate the $M(x)$ curve two more times to get $Y(x)$. That implies that if $M(x)$ is linear (a polynomial of order one), then the shape $Y(x)$ of the deflected beam is a polynomial of order three. And if $M(x)$ is quadratic (a polynomial of order two) then the shape $Y(x)$ of the deflected beam is a polynomial of order four. So it turns out that the maximum deflection of a beam of length L under a concentrated load is usually proportional to L^3, and the maximum deflection of a beam of length L under a uniform distributed load is usually proportional to L^4, just because of calculus.

(Onouye/Kane ch8: bending and shear stresses in beams)

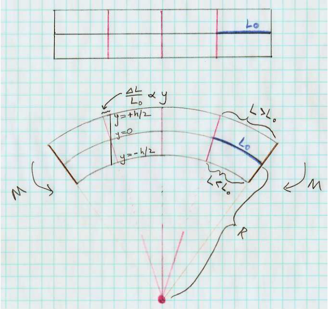
The *neutral axis* of a beam's cross-section lies along the vertical centroid $\overline{y}$ of the cross-section. Extending the neutral axis along the length of the beam defines the *neutral surface*. If certain conditions are met (the beam is initially straight, is of constant cross-section, and is of uniform composition; the beam is elastic and has equal elastic moduli in tension and compression; the beam is bent only with couples (bending moments at the ends); the beam is not twisted), then the longitudinal elements (fibers — easy to imagine for a wooden beam) of the neutral surface will be neither in tension nor in compression; they will undergo no change in length.

For a "simply supported" beam (which makes a  shape under load), longitudinal fibers below the neutral surface are in tension (elongated), while fibers above the neutral surface are in compression (shortened). For a cantilever (which makes a  shape under load), fibers above the neutral surface are in tension, while fibers below the neutral surface are in compression. It helps to imagine a wooden beam with fibers (grains) running along the axial length of the beam.

Let's imagine an initially horizontal beam of length L_0 bent into a  shape by applying a bending moment M at each end: counterclockwise at the left end and clockwise at the right end. Fibers above the neutral axis ($y > 0$) will be lengthened ($L > L_0$) while fibers below the neutral axis ($y < 0$) will be shortened ($L < L_0$).

A key idea is that we can approximate the deflected beam as an arc of a circle of radius R, where the bending moment is inversely related to the radius of curvature of the beam: $M \propto 1/R$. The larger the bending moment, the tighter the circular arc into which the beam bends. For a constant bending moment M, lines that are initially vertical converge toward the center of the circle, as shown below.

We can use similar triangles to argue that the longitudinal strain, $\delta L/L_0$, is proportional to the distance above the neutral surface. More precisely, $\Delta L/L_0 = y/R$.



Next, use the symbol e to denote the axial strain $\Delta L/L_0$, and use the symbol f to denote stress, which is force/area. For an elastic material, $f = eE$, where E is Young's modulus. So we have $y/R = \Delta L/L_0 = e = f/E$. So the axial stress (force per unit area) exerted by the fibers a distance y above the neutral axis is $f = Ey/R$.

Now, using the language of calculus, consider an infinitesimal fiber of area $\mathrm{d}A$ a distance y above the neutral surface. Using a pivot along the neutral axis, the torque (bending moment) exerted by the longitudinal fiber of area $\mathrm{d}A$ equals force times lever arm. The force is $\mathrm{d}F = f\,\mathrm{d}A$ (stress times area) and the lever arm is y. So the infinitesimal bending moment exerted by this infinitesimal fiber is

$$\mathrm{d}M = y\,\mathrm{d}F = y\,f\,\mathrm{d}A = y\,\left(\frac{Ey}{R}\right)\,\mathrm{d}A = \frac{E}{R}\,y^2\,\mathrm{d}A.$$

So the bending moment M exerted by a curved beam is

$$M = \int \mathrm{d}M = \frac{E}{R} \int y^2 \, \mathrm{d}A = \frac{EI}{R} \tag{1}$$

where R is the curved beam's radius of curvature, and $I = \int y^2 \, \mathrm{d}A$ is the "second moment of area" introduced in chapter 6.

Using $f = Ey/R$ to eliminate R, we can also write

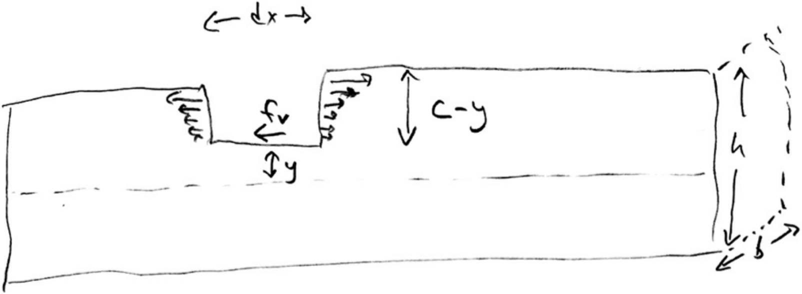
$$f = My/I, \tag{2}$$

or using c to represent the most extreme value of $|y|$ (for the fibers farthest from the neutral surface), the maximum bending stress is $\boxed{f_{\max} = Mc/I}$. After drawing the moment diagram $M(x)$, you can use the maximum value of $|M(x)|$ and your knowledge of the beam cross-section to determine the maximum bending stress $f_{\max}$, which can be compared with the allowable stress F_{allow} for the material of which the beam is composed.

Since I and c are just geometrical properties of the beam cross-section, their ratio is given a name: $\boxed{S = I/c}$ is called the *section modulus*, where I is second moment of area (w.r.t. the neutral axis) and c is the distance from the neutral surface to the top or bottom of the beam (whichever is larger, if the beam is asymmetric). We can then write the bending stress in the extreme fibers as $\boxed{f_b = M/S}$. Alternatively, if you are working with material of a given allowable bending stress F_b and the maximum (in absolute value) bending moment for your loading conditions is M_{max}, then you need to choose a cross-section for your beam whose section modulus is larger than $\boxed{S_{\mathrm{required}} = M_{\mathrm{max}}/F_b}$. For standard beam shapes, values of section modulus S are tabulated. The dimensions of section modulus are length^3, e.g. cubic meters, cubic centimeters, or cubic inches.

(Need diagram.) Imagine a fiber located a height y above the neutral surface. At position x along the length of the beam, the axial (bending) stress in this fiber will be $f_b = My/I$, using equation (2). Because $M(x)$ varies along the length of the beam, this bending stress will vary with x:

$$\frac{\mathrm{d}f_b}{\mathrm{d}x} = \frac{y}{I}\,\frac{\mathrm{d}M(x)}{\mathrm{d}x} = \frac{y}{I}\,V(x)$$



(Need a nicer diagram.) Now imagine the forces acting on a rectangular block of beam that extends longitudinally from x to $x + \mathrm{d}x$, extends vertically from y to c (measured from the neutral surface, where c is the top surface of the beam), and extends the entire width b of the beam cross-section. Since stress = force/area, each force is the integral of stress over the corresponding area. The horizontal force acting on the left surface of the block is $\int_y^c f_b(x)\, b\, \mathrm{d}y$. The horizontal force acting on the right surface of the block is $\int_y^c f_b(x + \mathrm{d}x)\, b\, \mathrm{d}y$.

Along the top surface there is no force, as there is no material above the top of the beam. But acting horizontally along the bottom surface of the rectangular block is the *shear stress*, f_v. The corresponding force is $f_v\, b\, \mathrm{d}x$. The horizontal forces on these three surfaces must sum arithmetically to zero:

$$f_v\, b\, \mathrm{d}x = \int_y^c [f_b(x+\mathrm{d}x) - f_b(x)]\, b\, \mathrm{d}y = \int_y^c [\frac{\mathrm{d}f_b}{\mathrm{d}x}\, \mathrm{d}x]\, b\, \mathrm{d}y$$

$$= \int_y^c [\frac{y}{I}\, V(x)\, \mathrm{d}x]\, b\, \mathrm{d}y.$$

We can cancel $\mathrm{d}x$, and for the special case of a rectangular cross-section (so b is independent of y) we can cancel b, replace c with $h/2$, and replace I with $bh^3/12$:

$$f_v = \frac{V(x)}{I} \int_y^{h/2} y\, \mathrm{d}y = \frac{V}{I}\left[\frac{h^2}{8} - \frac{y^2}{2}\right] = \frac{12V}{bh^3}\left[\frac{h^2}{8} - \frac{y^2}{2}\right]$$

$$= \frac{3V}{2A}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

where $A = bh$ is the area of the beam cross-section.

The maximum shear stress is $\frac{3}{2}V/A$ (for a rectangular cross-section) and occurs at the neutral surface ($y = 0$) at the longitudinal position x where the shear force $|V(x)|$ is largest — which usually occurs at the supports.

To envision shear strain (which by Hooke's law is proportional to shear stress), bend a deck of cards into a $\smile$ shape and observe how each card slides against its neighbors.

In many circumstances, building codes will specify the maximum allowable deflection of a beam of length L as some small fraction of the length of the beam: for example, an $L/360$ deflection limit would imply that a horizontal beam of length $3.6\,\mathrm{m}$ can deflect no more than $1\,\mathrm{cm}$ vertically under load. We use the symbol Δ to indicate the vertical deflection of the beam. A positive value of Δ points downward, in the $-y$ direction. We can consider the deflection $\Delta(x)$ as a function of horizontal position x along the length of the beam, or we can consider the maximum deflection $\Delta_{\max}$. We want to be able to evaluate $\Delta_{\max}$ for a hypothetical beam under load and impose an allowable deflection criterion, for example $\Delta_{\text{max}} \leq L/360$.

Solving equation (1) for R, the radius of curvature of a loaded beam is $R = EI/M$. The beam is straighter (larger R) when the elastic modulus E and second moment of area I are larger; the beam curves more (smaller R) when the bending moment M is larger. The radius of curvature R of a function $y = f(x)$ is given in calculus by the formula

$$\frac{1}{R} = \frac{y''}{(1 + (y')^2)^{3/2}} \approx y''.$$

We know that the second derivative of a function is related to its curvature: if $y'' = 0$ then the function is a straight line (no curvature); if $y'' > 0$ then the function has "concave up" curvature; and if $y'' < 0$ then the function has "concave down" curvature. In architectural structures, one deals with beams whose slope is very small: $|y'| \ll 1$, meaning that the slope of the beam under load is much smaller than one radian. (A radian is 57.3°, which would be a very large slope for a deflected beam.) So it is conventional to use the small-angle ($|y'| \ll 1$) approximation: $y'' \approx 1/R$.

In the small-angle approximation, the second derivative $\Delta''(x)$ of the deflected beam shape $\Delta(x)$ obeys the *Euler-Bernoulli beam equation*

$$-\Delta''(x) = \frac{1}{R} = \frac{M}{EI}.$$

The minus sign is because $\Delta(x)$ increases in the $-y$ direction. We can integrate the bending-moment curve $M(x)$ twice to get the deflected shape $\Delta(x)$ of the beam:

$$-\Delta(x) = \frac{1}{EI} \int \mathrm{d}x \int M(x)\,\mathrm{d}x$$

Since the moment curve $M(x)$ is usually a quadratic curve for a beam with a uniform distributed load w and is usually a piecewise linear curve for a beam with a concentrated load P, it makes sense that $\Delta(x)$ is usually a fourth-order polynomial for a uniformly loaded beam and is usually a cubic polynomial for a concentrated load. The most common cases are tabulated in books and online references.

For example, a simply supported beam has $\Delta_{\max} = 5wL^4/(384EI)$ for uniform load w or $\Delta_{\max} = PL^3/(48EI)$ for a concentrated load P at mid-span. A cantilever has $\Delta_{\max} = wL^4/(8EI)$ for uniform load w and $\Delta_{\max} = PL^3/(3EI)$ for concentrated load P at the free end. You can look up many more specific cases.

Here's where the crazy $5/384$ comes from: A simply supported beam of length L and uniform load w has shear curve $V(x) = (\frac{1}{2}L - x)w$ and bending moment curve $M(x) = (Lx - x^2)w/2$. Integrating twice,

$$\Delta(x) = -\frac{1}{EI}\int \mathrm{d}x \int M(x)\,\mathrm{d}x = -\frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12} + C_1 x + C_2\right)$$

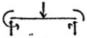
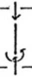
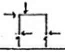
The boundary condition $\Delta(0) = 0$ gives $C_2 = 0$ and $\Delta(L) = 0$ gives $C_1 = -L^3/12$. So $\Delta(x) = \frac{w}{2EI}\left(\frac{x^4}{12} - \frac{Lx^3}{6} + \frac{L^3 x}{12}\right)$. Plugging in $x = L/2$ (which is where $\Delta'(x) = 0$) gives $\Delta_{\max} = 5wL^4/(384EI)$. To get the two integration constants for a simply supported beam, use $\Delta(0) = \Delta(L) = 0$. For a cantilever whose left end is fixed, the integration constants would instead be given by $\Delta(0) = 0$ and $\Delta'(0) = 0$.

Beam design criteria usually include the following:

- Axial stress in the extreme fibers of the beam (farthest from the neutral surface) must be smaller than the allowable bending stress, F_b, which depends on the material (wood, steel, etc.) Maximum bending stress happens where bending moment $|M(x)|$ is largest.
- Shear stress, in both y ("transverse") and x ("longitudinal") directions, must be smaller than the allowable shear stress F_v, which also depends on the material (wood, steel, etc.). Shear stress is maximum where $|V(x)|$ is largest, and is largest near the neutral surface.
- The above two are "strength" criteria. A third condition is a "stiffness" criterion: The maximum deflection under load must satisfy the building code: typically $\Delta_{\max} < L/360$, though in some cases the denominator is smaller, e.g. 120, 180, 240. For a uniform load, the maximum deflection occurs farthest away from the supports. If deflection is too large, plaster ceilings develop cracks, and floors feel uncomfortably bouncy or sloped.

- Onouye/Kane also mention buckling as a beam failure mode. For a simply supported beam, the top is in compression while the bottom is in tension; vice-versa for a cantilever. In very deep beams (i.e. very tall in cross-section), the compression side can buckle or deflect sideways. Wood framing addresses this issue with sheathing (a.k.a. furring or strapping) nailed at close spacing perpendicular to the floor joists and solid blocking to prevent buckling at the ends. In a very deep I-beam, the flange on the compression side is susceptible to buckling.

ELEMENTS OF STRUCTURE

	1D in 1D	Beam
	1D in 1D	Column
	1D in 2D	Frame
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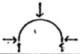
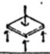
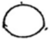
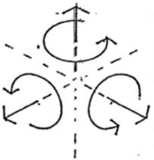
	1D in 2D	Arch
	2D in 2D	Slab Plate
	2D in 2D	Wall
	1D in 3D	Spaceframe
	2D in 3D	Vault Shell
	2D in 3D	Dome Membrane
	3D in 3D	Foundations Soils

DEGREES OF FREEDOM

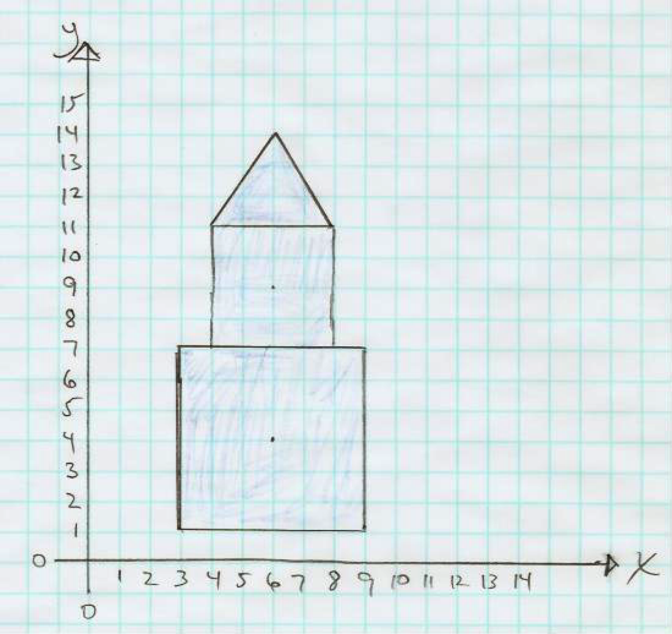


IDEAS OF STRUCTURE

Force
Restraint
Equilibrium
Stress
Deformation
Elasticity
Geometry
Strength
Stiffness
Scale
Continuity
Stability
Safety

- The idea of computing centroids of simple and composite shapes is very, very briefly introduced in O/K ch3 (in the context of "distributed loads"), and is discussed in much more detail in O/K ch6 (cross-sectional properties).
- Let's go through one example using rectangles and triangles. It will help you in cases when you need to solve for the "reaction forces" on a beam that carries distributed loads. (Example coming up next.)

What is X_{centroid} for the shaded area?



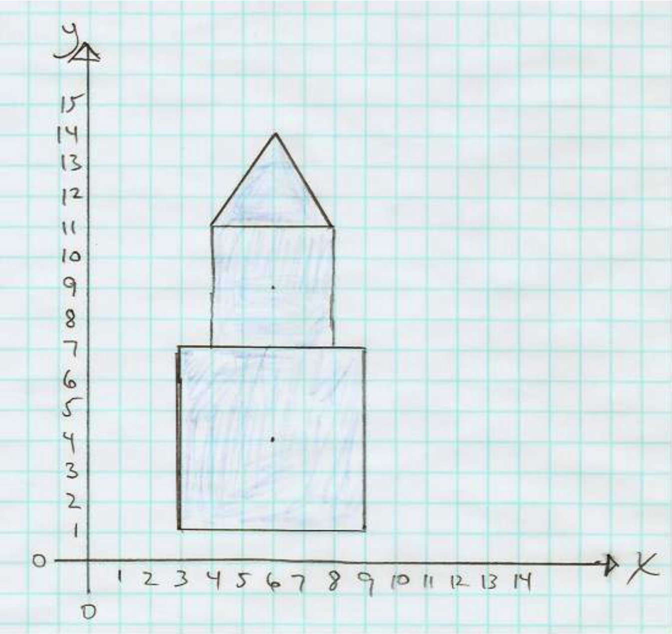
(A) 0

(B) 3

(C) 6

(D) 9

What are the areas of the three individual polygons?



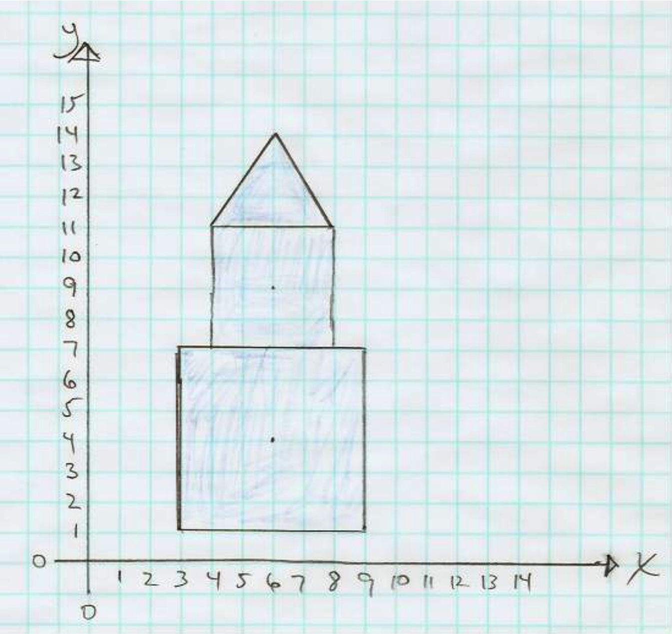
(A) 36, 16, 16

(B) 36, 16, 12

(C) 36, 16, 8

(D) 36, 16, 6

What are the Y_{centroid} values of the three individual polygons?



(A) 4, 9, 11

(B) 4, 9, 11.667

(C) 4, 9, 12

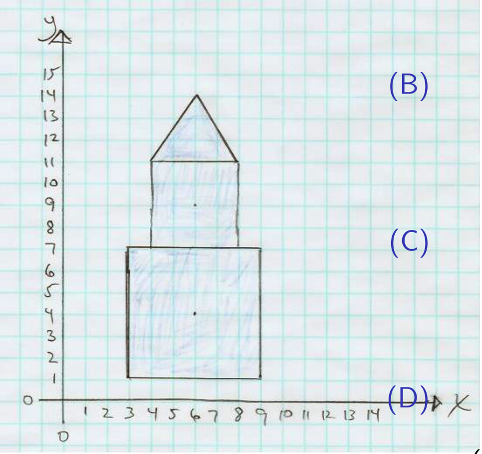
(D) 4, 9, 12.333

(E) 4, 9, 12.5

(F) 4, 9, 13

(G) 4, 9, 14

What is Y_{centroid} for the whole shaded area?



(A)

$$\frac{4+9+12}{3} = 8.33$$

(B)

$$\frac{(4)(36)+(9)(16)+(12)(6)}{36+16+6} = 6.21$$

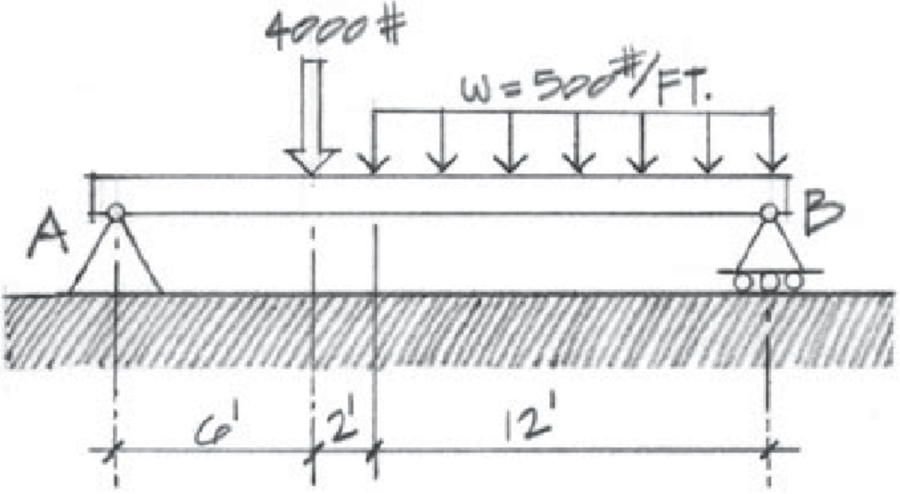
(C)

$$\frac{(4)(36)+(9)(16)+(12)(6)}{4+9+12} = 14.4$$

(D)

$$\frac{(4^2)(36)+(9^2)(16)+(12^2)(6)}{36+16+6} = 47.2$$

Last week, you tried one problem similar to this (but using metric units): Determine the support reactions at A and B.



```
ClearAll["Global`*"];
load1Force = 4000.0 pound;
load1X = 6.0 foot; Measure positions w.r.t. support A;
load2Force = (500.0 pound / foot) * (12 foot)
```

```
6000. pound
```

```
Find centroid of distributed load, for equivalent concentrated load;
load2X = (6.0 + 2.0 + 12.0 / 2) foot
```

```
14. foot
```

```
Evaluate moments about pivot A;
Solve[0 == By (6.0 + 2.0 + 12.0) foot - load1Force * load1X -
   load2Force * load2X, By]
```

```
{{By → 5400. pound}}
```

```
By = By /. First[%]
```

```
5400. pound
```

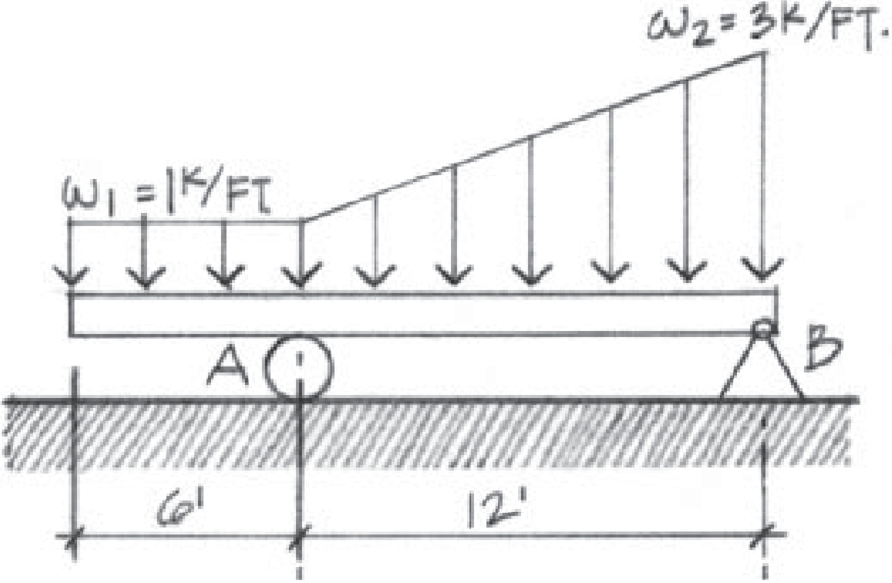
```
Solve[0 == Ay + By - load1Force - load2Force, Ay]
```

```
{{Ay → 4600. pound}}
```

```
Solve[0 == Ax]
```

```
{{Ax → 0}}
```

This one is harder, because the distributed load is non-uniform:
Determine the support reactions at A and B.



```
ClearAll["Global`*"];
foot = Quantity[1.0, "foot"];
pound = Quantity[1.0, "pound"];
Rectangle for uniform 1k/foot load spanning entire beam.;
load1X = 0.5 (6 foot + 12 foot)
```

```
9. ft
```

```
load1Force = (1000 pound / foot) * (18.0 foot)
```

```
18 000. lb
```

```
Trianglular load that sits above uniform load.;
load2X = 18.0 foot - (12.0 foot) / 3
```

```
14. ft
```

```
load2Force = 0.5 * (12.0 foot) * (2000 pound / foot)
```

```
12 000. lb
```

```
Moments about B;
L = 18.0 foot;
Solve[
 0 == load1Force * (L - load1X) + load2Force * (L - load2X) -
   Ay * (12 foot), Ay]
```

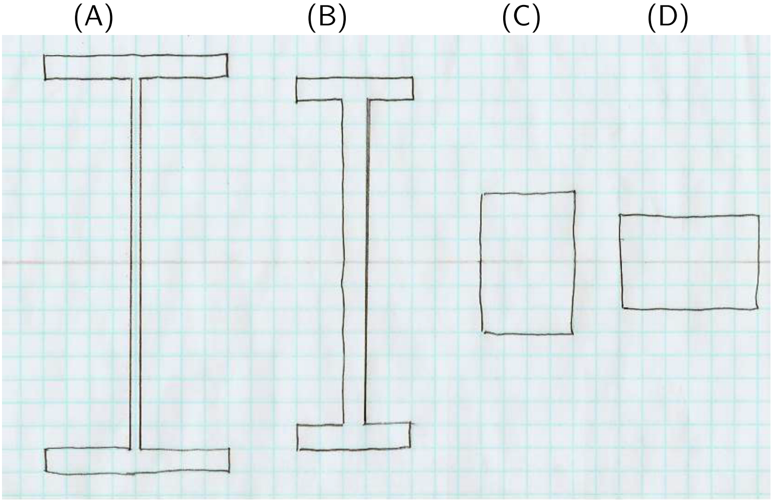
```
{{Ay → 17 500. lb}}
```

```
Ay = Ay /. First[%]
```

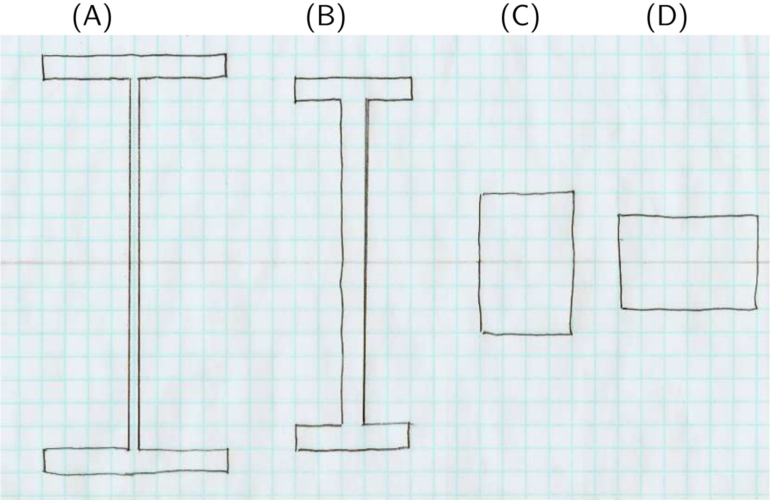
```
17 500. lb
```

```
Solve[0 == Ay + By - load1Force - load2Force, By]
```

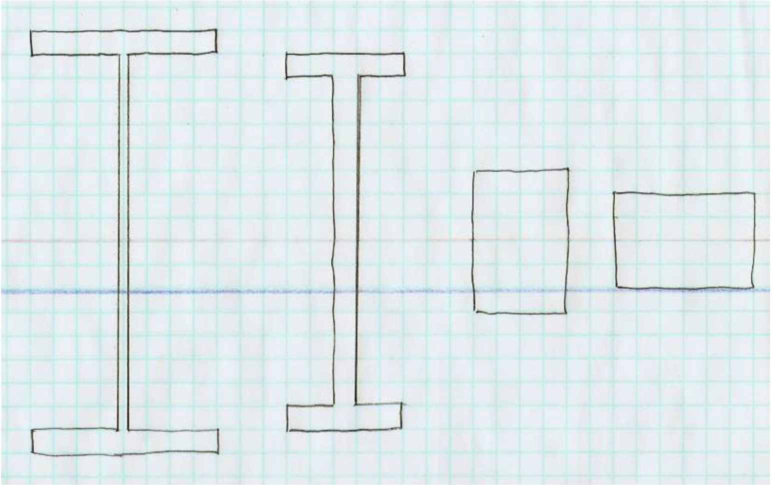
```
{{By → 12 500. lb}}
```



Each shape has the same area: 24 squares. Which shape has the largest $I_x = \int y^2 \mathrm{d}A$ ("second moment of area about the x-axis"), with $y = 0$ given by the faint horizontal red line at the center?



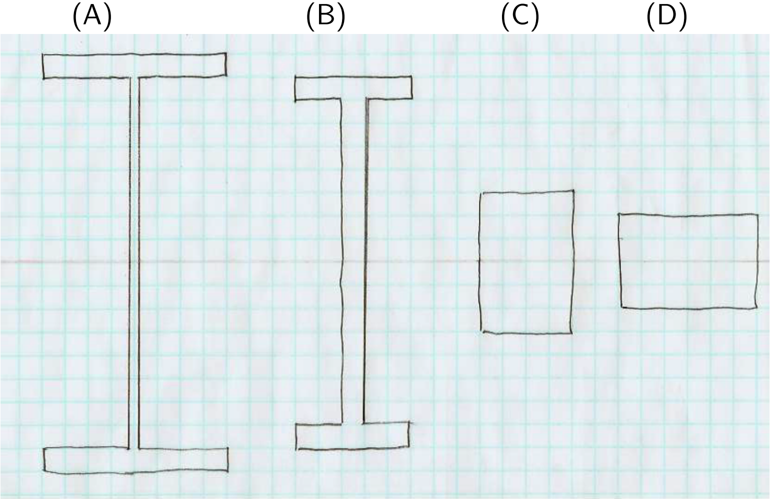
Each shape has the same area: 24 squares. Which shape has the **smallest** $I_x = \int y^2 \mathrm{d}A$ ("second moment of area about the x-axis"), with $y = 0$ given by the faint horizontal red line at the center?



If you moved the x-axis down by a couple of grid units, what would happen to $I_x = \int y^2 \mathrm{d}A$ for each shape? Would I_x change? Would I_x change by the same amount for each shape?

(A) yes (B) no

(Think: "parallel-axis theorem.")



Given that $I_x = \int y^2 \mathrm{d}A = \frac{1}{12} bh^3$ for a rectangle centered at $y = 0$, let's use the parallel-axis theorem to calculate I_x for shapes A, B, C, and D. For definiteness, let each graph-paper box be $1 \text{ cm} \times 1 \text{ cm}$. So the units will be cm^4.

Let's do the two rectangular shapes first, since they're quick.

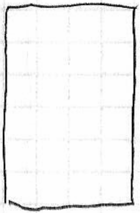
Then, the trick for the non-rectangular shapes is to use (from O/K §6.3) the "parallel-axis theorem:"

$$I_x = \sum I_{xc} + \sum A\, d_y^2$$

where each sum is over the simple shapes that compose the big shape.

- I_{xc} is the simple shape's own I_x value about its own centroid (which is $bh^3/12$ for a rectangle),
- A is the simple shape's area, and
- d_y is the vertical displacement of the simple shape's centroid from $y = 0$ (which should be the centroid of the big shape).

(C)



$b = 4\,cm$

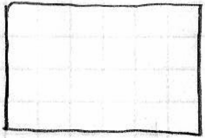
$h = 6\,cm$

$A = 24\,cm^2$

$y_c = 0$

$\frac{1}{12} bh^3 = \boxed{72\,cm^4}$

(D)



$b = 6\,cm$

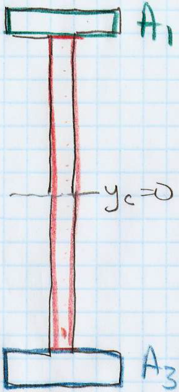
$h = 4\,cm$

$A = 24\,cm^2$

$y_c = 0$

$\frac{1}{12} bh^3 = \boxed{32\,cm^4}$

(B)



$A_1 = 5\,cm^2$ $b_1 = 5\,cm$ $h_1 = 1\,cm$

$y_{c_1} = +7.5\,cm$

$A_2 = 14\,cm^2$ $b_2 = 1\,cm$

$h_2 = 14\,cm$

$y_{c_2} = 0$

$A_3 = 5\,cm^2$ $b_3 = 5\,cm$ $h_3 = 1\,cm$

$y_{c_3} = -7.5\,cm$

$\frac{1}{12} b_1 h_1^3 = 0.417\,cm^4$

$A_1 y_{c_1}^2 = 281.25\,cm^4$

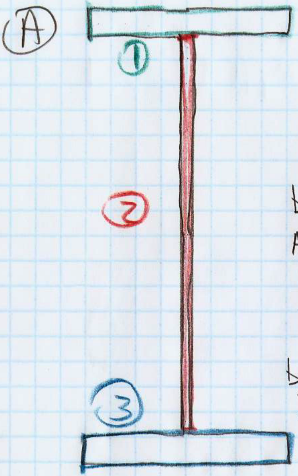
$\frac{1}{12} b_2 h_2^3 = 228.67\,cm^4$

$A_2 y_{c_2}^2 = 0$

$\frac{1}{12} b_3 h_3^3 = 0.417\,cm^4$

$A_3 y_{c_3}^2 = 281.25\,cm^4$

$$I_B = \frac{1}{12} b_1 h_1^3 + \frac{1}{12} b_2 h_2^3 + \frac{1}{12} b_3 h_3^3 + A_1 y_{c_1}^2 + A_2 y_{c_2}^2 + A_3 y_{c_3}^2 = \boxed{792\,cm^4}$$



$b_1 = 8\,cm \quad h_1 = 1\,cm$

$A_1 = 8\,cm^2 \quad y_{c_1} = +8.5\,cm$

$\frac{1}{12} b_1 h_1^3 = 0.67\,cm^4$

$A_1\, y_{c_1}^2 = 578\,cm^4$

$b_2 = 0.5\,cm \quad h_2 = 16\,cm$

$A_2 = 8\,cm^2 \quad y_{c_2} = 0$

$\frac{1}{12} b_2 h_2^3 = 170.67\,cm^4$

$A_2\, y_{c_2}^2 = 0$

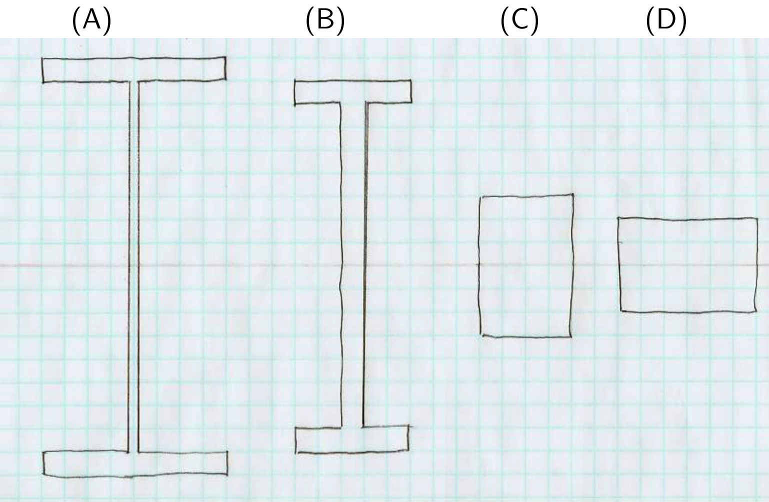
$b_3 = 8\,cm \quad h_3 = 1\,cm$

$A_3 = 8\,cm^2 \quad y_{c_3} = -8.5\,cm$

$\frac{1}{12} b_3 h_3^3 = 0.67\,cm^4$

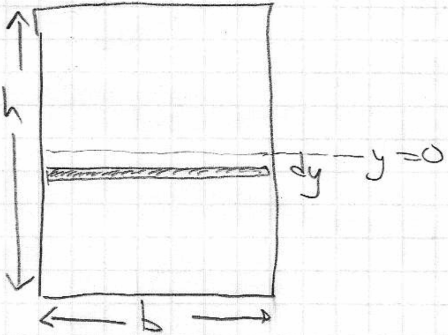
$A_3\, y_{c_3}^2 = 578\,cm^4$

$$I_A = \frac{1}{12} b_1 h_1^3 + \frac{1}{12} b_2 h_2^3 + \frac{1}{12} b_3 h_3^3 + A_1\, y_{c_1}^2 + A_2\, y_{c_2}^2 + A_3\, y_{c_3}^2 = \boxed{1328\,cm^4}$$



Each shape has same area $A = 24\ \mathrm{cm}^2$, but "second moment of area" is $I_A = 1328\ \mathrm{cm}^4$, $I_B = 792\ \mathrm{cm}^4$, $I_C = 72\ \mathrm{cm}^4$, $I_D = 32\ \mathrm{cm}^4$. That's the motivation for the "I" shape of an I-beam: to get a large "second moment of area," $I = \int y^2\,\mathrm{d}A$. The deflection of a beam under load is inversely proportional to I.

rectangle

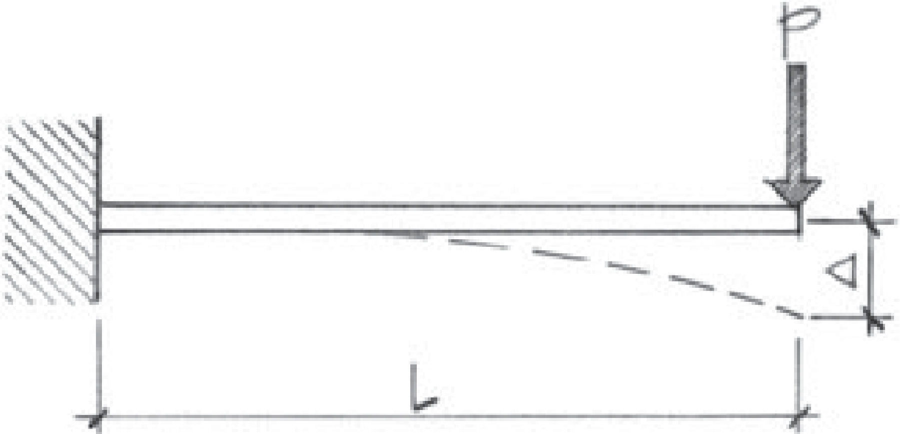


$dA = b\,dy$

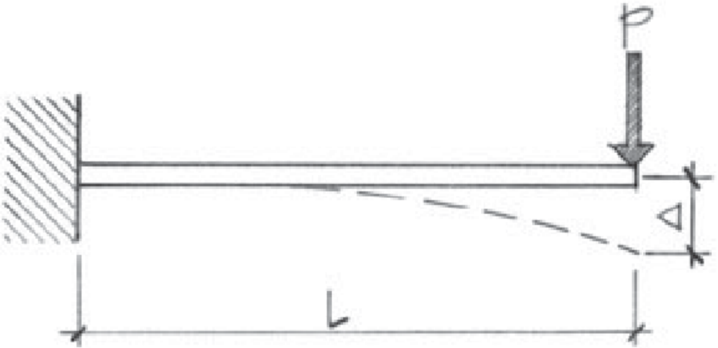
$$I = \int y^2\,dA = \int_{y=-\frac{h}{2}}^{y=h/2} y^2\,b\,dy = \left[\frac{by^3}{3}\right]_{y=-\frac{h}{2}}^{h/2}$$

$$= \frac{b(h/2)^3}{3} - \frac{b(-h/2)^3}{3} = \frac{bh^3}{12}$$

We can use the Method of Sections to study the internal forces and torques ("moments") within a beam. Consider this cantilever beam (whose own weight we neglect here) supporting a concentrated "load" force P at the far end. The left half is what holds up the right half. What force and torque ("moment") does the left half exert on the right half? Does the answer depend on where we "section" the beam?

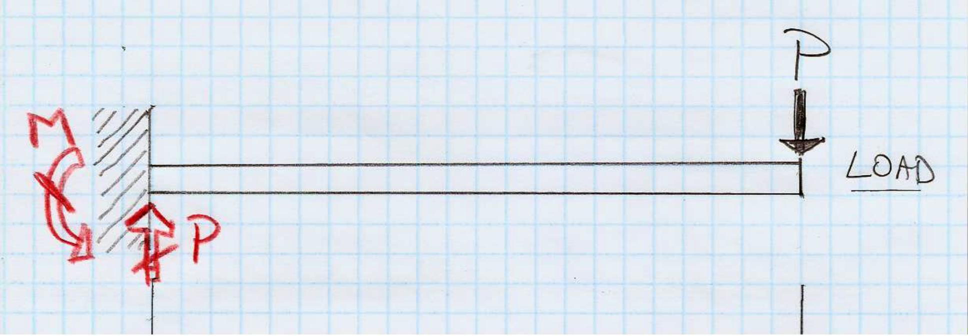


We draw "load diagram" (basically a FBD for the beam), then the "shear (V) diagram" below that, then the "moment (M) diagram" below that. Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.

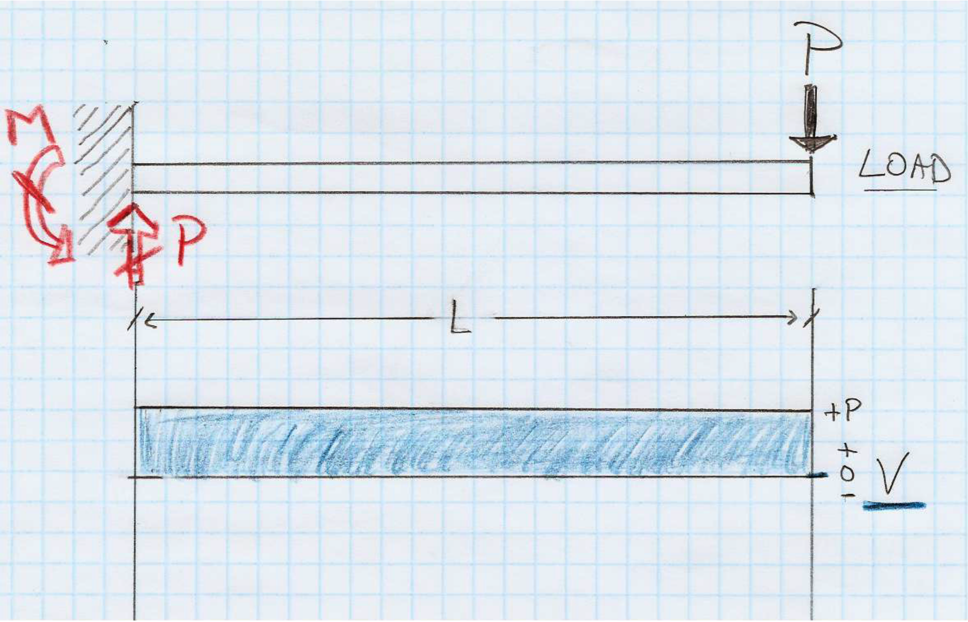


Another way to state the $V(x)$ sign convention: $V(x)$ is the running sum of all (upward minus downward) forces exerted on the beam, from the left side up to and including x.

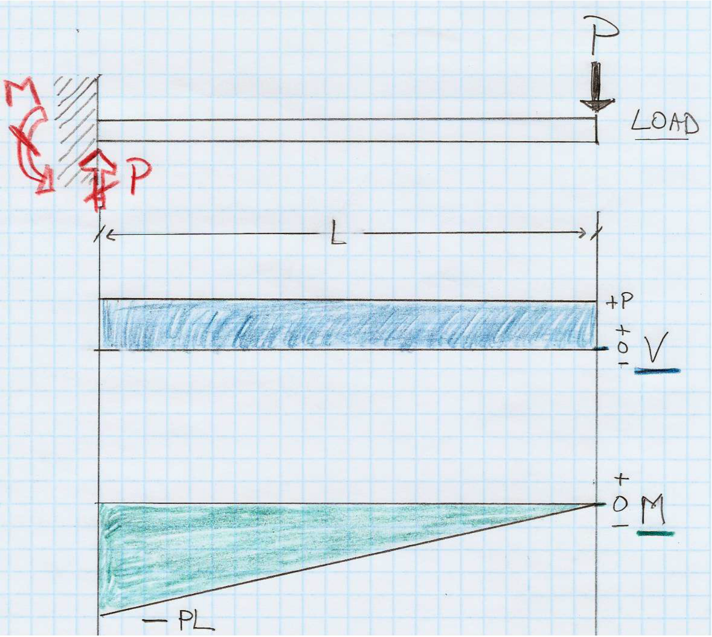
We draw "load diagram" (basically a FBD for the beam), then the "shear (V) diagram" below that, then the "moment (M) diagram" below that. Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.



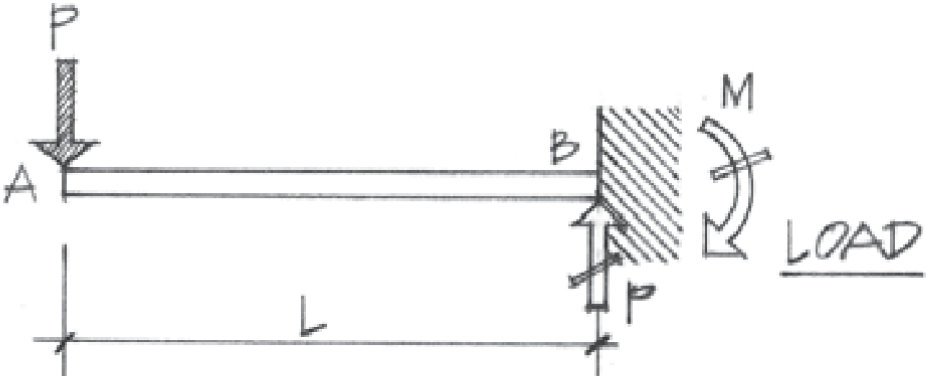
Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.



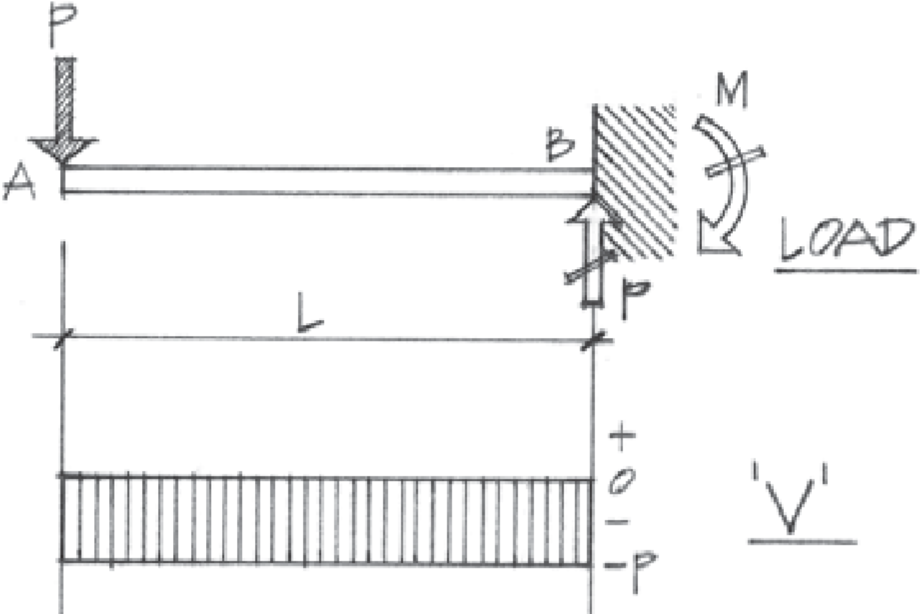
Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.



Let's try a mirror image of the same cantilever beam. Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.



Sign conventions: $V > 0$ when beam LHS section is pulling up on beam RHS; $M > 0$ when beam is smiling.



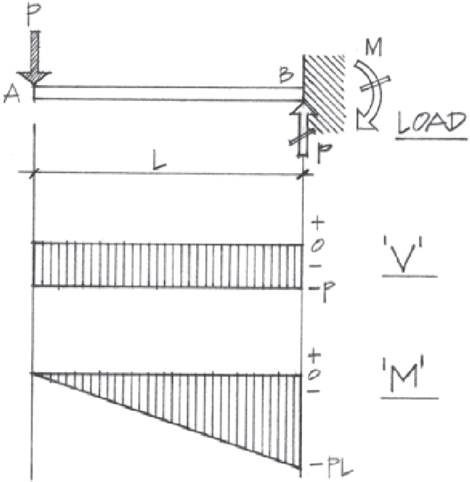
Sign conventions: $V(x) > 0$ when beam left of x is pulling up on beam right of x. $M(x) > 0$ when beam is smiling.

Transverse shear $V(x)$ is the running sum of forces on beam, from $0 \ldots x$, where upward = positive.

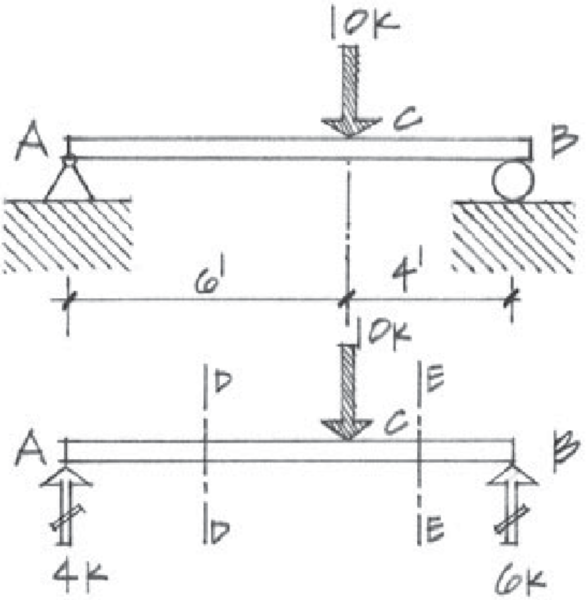
Bending moment $M(x)$ is the torque exerted by each side of the beam, cut at x, on the other side; but beware of sign convention.

$$V(x) = \frac{\mathrm{d}}{\mathrm{d}x} M(x)$$

The V diagram graphs the slope of the M diagram.



Draw V and M for this "simply supported" beam: $V(x)$ is running sum (up − down) of forces on beam. $M > 0$ when beam smiles.

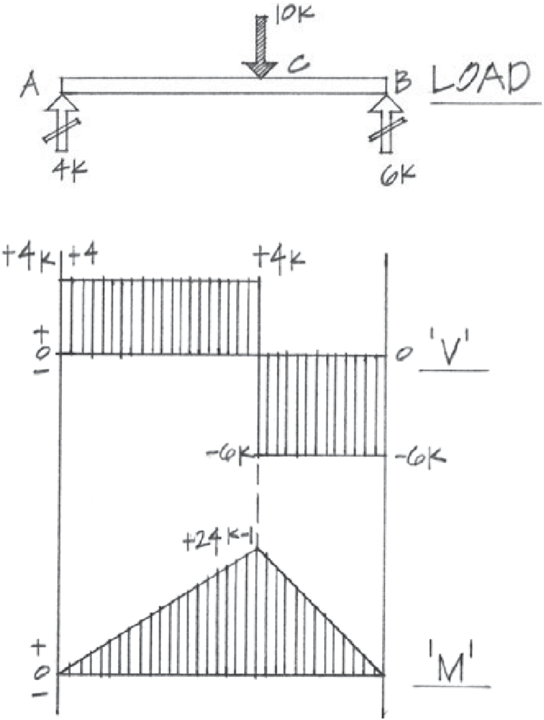


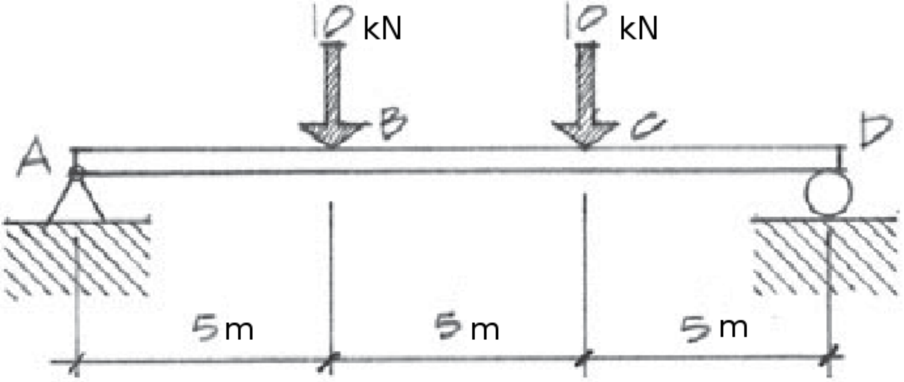
$$V(x) = \frac{\mathrm{d}}{\mathrm{d}x} M(x)$$

The shear (V) diagram equals the slope of the moment (M) diagram.

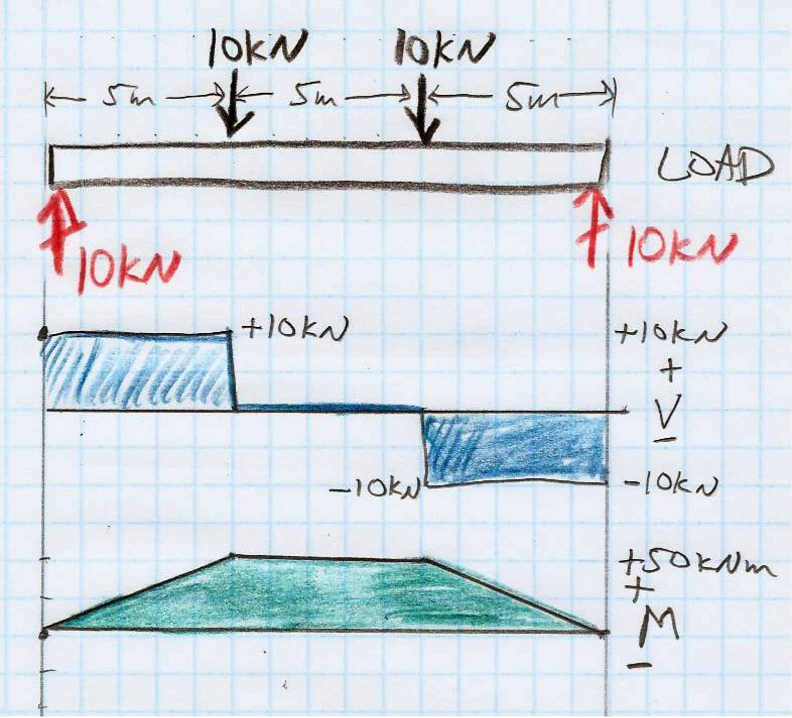
$$M(x) = \int V(x)\,\mathrm{d}x$$

But be careful about the M values at the ends — depends how the beam is supported. A free, hinged, or roller-supported end has $M = 0$: support exerts no torque on that end. Fixed end of cantilever has $M \neq 0$.





Let's try drawing load, V, and M diagrams for this simply-supported beam. Pretend the units are meters and kilonewtons rather than the original drawing's feet and kilopounds ("kips").



10kN
10kN
5m
5m
5m
LOAD
10kN
10kN
+10kN
+10kN
+
V
−
−10kN
−10kN
+50kNm
+
M
−

Shear (V) and moment (M) diagrams:

- First draw a "load diagram," which is an EFBD that shows all of the vertical forces acting on the beam.
- The "shear diagram" $V(x)$ graphs the running sum of all vertical forces (both supports and loads) acting on the beam, from the left side up to x, where upward = positive, downward = negative.
- To draw the "moment diagram" $M(x)$, note that V is the slope of M:

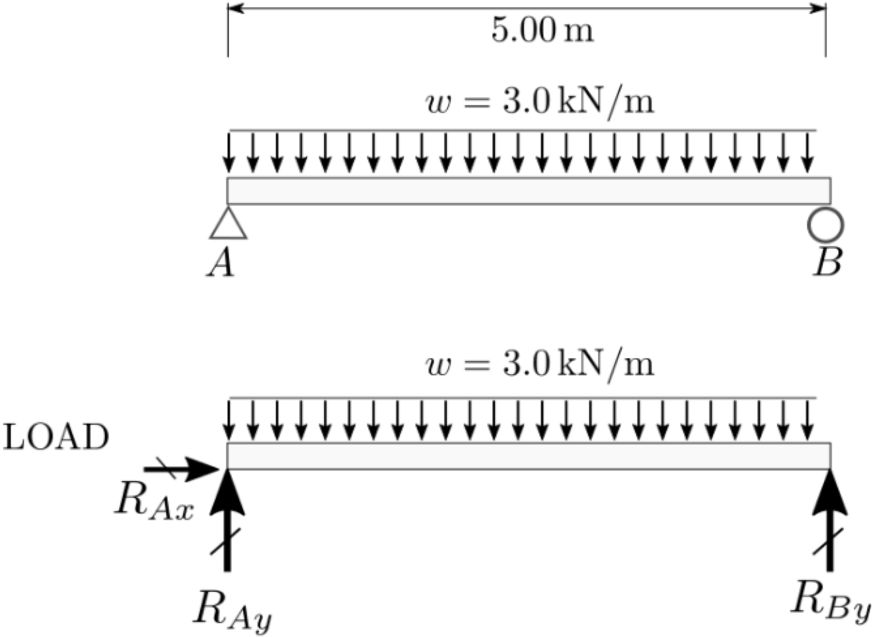
$$V(x) = \frac{\mathrm{d}}{\mathrm{d}x} M(x)$$

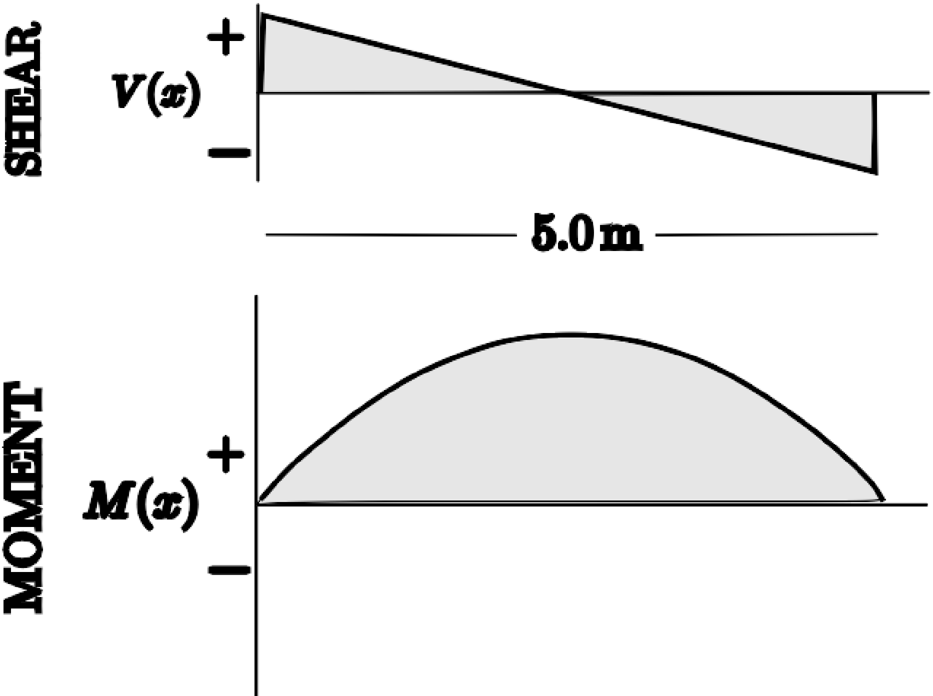
- The change in M from x_1 to x_2 is given by

$$M_2 - M_1 = (x_2 - x_1)\, V_{1\to 2}^{\mathrm{average}}$$

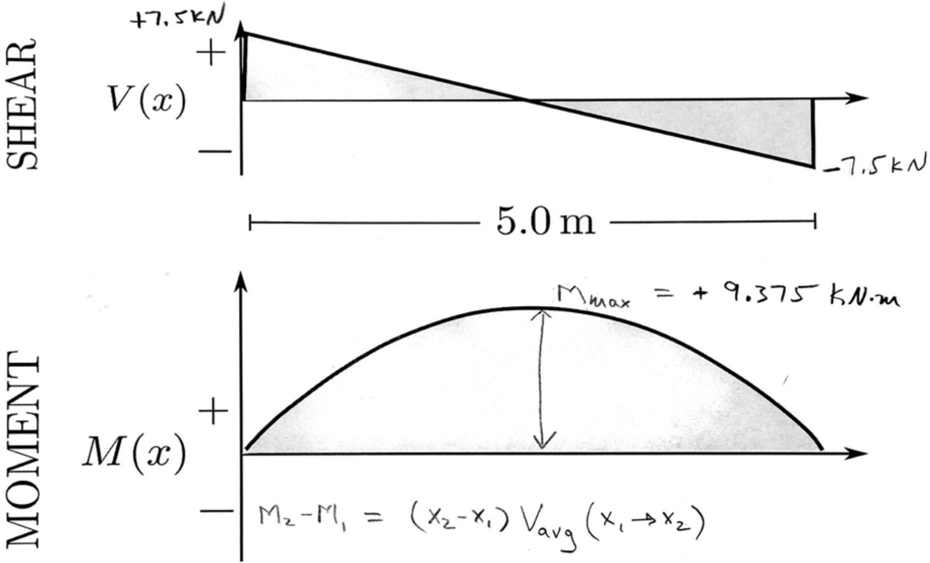
- If an end of a beam is unsupported ("free"), is hinge/pin supported, or is roller supported, then $M = 0$ at that end. You can only have $M \neq 0$ at an end if the support at that end is capable of exerting a torque on the beam — for example, the fixed end of a cantilever has $M \neq 0$.

Let's try drawing $V(x)$ and $M(x)$ diagrams for this simply supported beam with uniform distributed load:

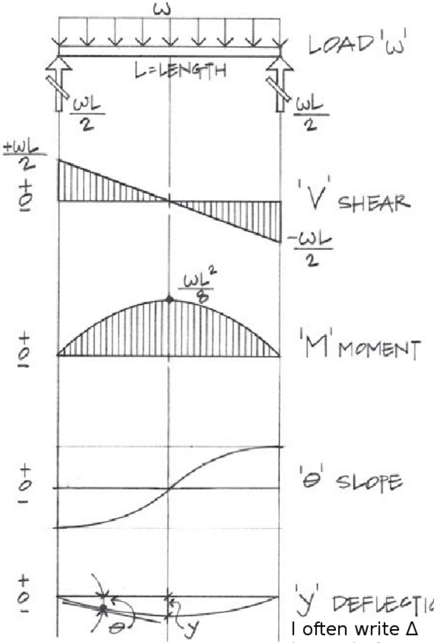




SHEAR
+
V(x)
−
5.0 m
MOMENT
+
M(x)
−



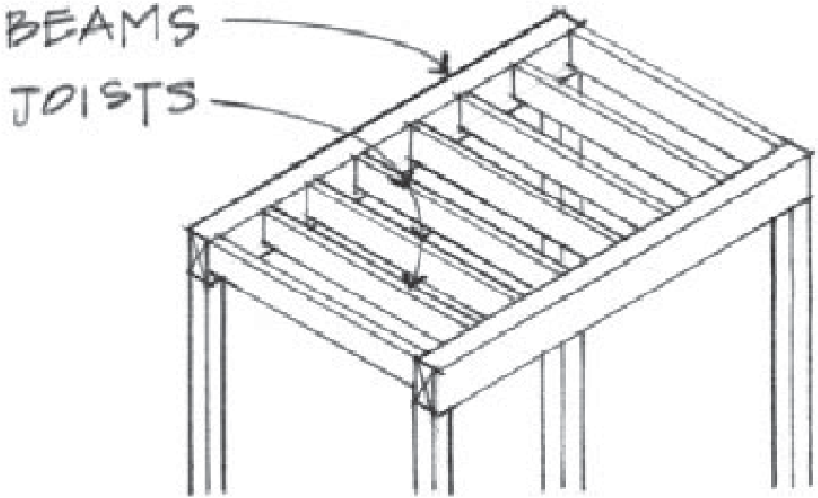
SHEAR
+7.5 kN
+
V(x)
−
−7.5 kN
5.0 m
MOMENT
Mmax = + 9.375 kN·m
+
M(x)
−
M2 − M1 = (x2 − x1) Vavg (x1 → x2)



I often write Δ instead of y

Figure 7.11 Relationship of load, shear, moment, slope, and deflection diagrams.

Why do we care about these beam diagrams, anyway? Usually the floor of a structure must carry a specified weight per unit area. The beams (beams, girders, joists, etc.) must be strong enough to support this load without failing and must be stiff enough to support this load without excessive deflection.



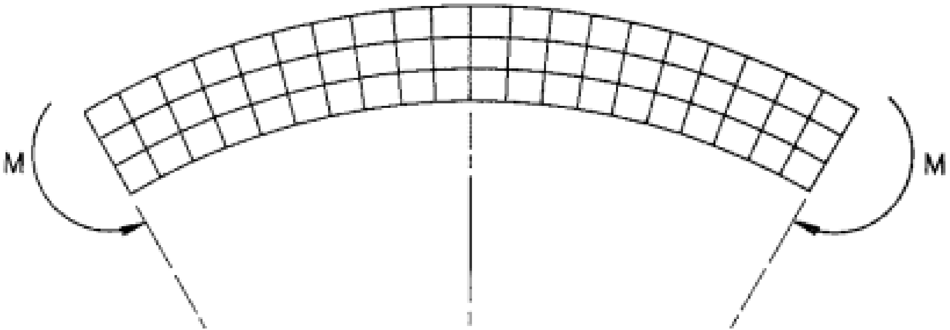
Beam criteria:

- Axial stress in the extreme fibers of the beam (farthest from neutral surface) must be smaller than the allowable bending stress, F_b, which depends on the material (wood, steel, etc.).
- This happens where $M(x)$ has largest magnitude.
- Shear stress (in both y ("transverse") and x ("longitudinal") must be smaller than the allowable shear stress, F_v, which is also a property of the material (wood, steel, etc.).
- This happens where $V(x)$ has largest magnitude, and (surprisingly) is largest near the neutral surface.
- The above two are "strength" criteria. The third one is a "stiffness" criterion:
- The maximum deflection under load must satisfy the building code: typically $\Delta y_{\mathrm{max}} < L/360$.
- For a uniform load, this happens farthest away from the supports. If deflection is too large, plaster ceilings develop cracks, floors feel uncomfortably bouncy or sloped.
- The book also notes buckling as a beam failure mode.

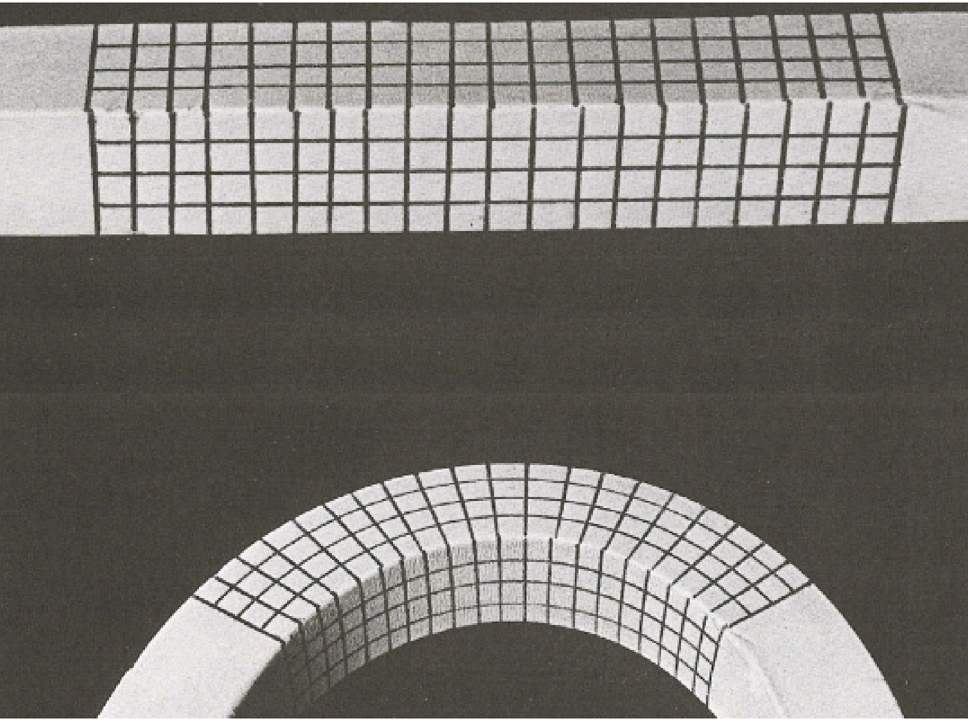
- Imagine (not so unrealistically) a wooden beam as a bunch of long parallel fibers glued together.
- The axial stress can't be so big that individual fibers will snap like strings, or crush like a cardboard box. Axial stress is largest where $M(x)$ is largest and at the fibers farthest from the neutral surface.
- The shear stress can't be so big that the "glue" fails to prevent adjacent fibers from sliding parallel to one another. Shear stress is largest where $V(x)$ is largest and (surprisingly) is largest near the neutral surface.
- The above two criteria relate to the **strength** of the material — its ability to resist **irreversible** damage. The third criterion will relate to the **stiffness** of the beam — its ability to resist **reversible** deflection that is uncomfortably large.

- The above two criteria related to the **strength** of the material — its ability to resist **irreversible** damage. The third criterion relates to the **stiffness** of the beam — its ability to resist **reversible** deflection that is uncomfortably large.
- If a beam deflects too much (bows into a $\smile$ shape), then plaster ceilings develop cracks, and floors feel uncomfortably bouncy or sloped. Building codes specify maximum deflection.
- Prof Farley points out that in typical cases, the deflection criterion turns out to be more strict than the strength criteria. So in practice, one first designs using the deflection criterion, then checks whether the strength criteria are satisfied, and iterates if necessary.

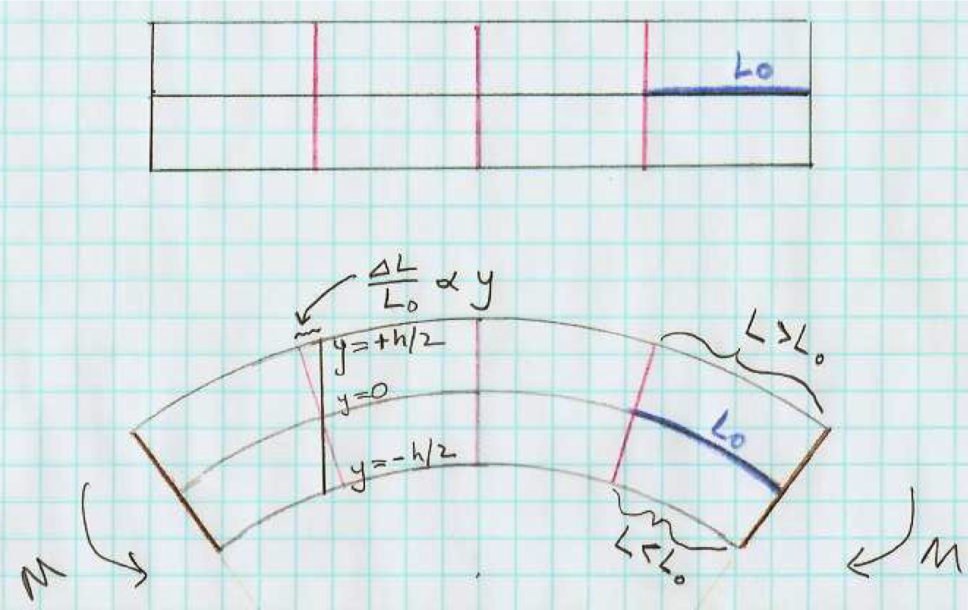
- Therefore, let's do some geometry and some math to try to figure out what determines how much a beam deflects into a curved shape when you put a load on it.



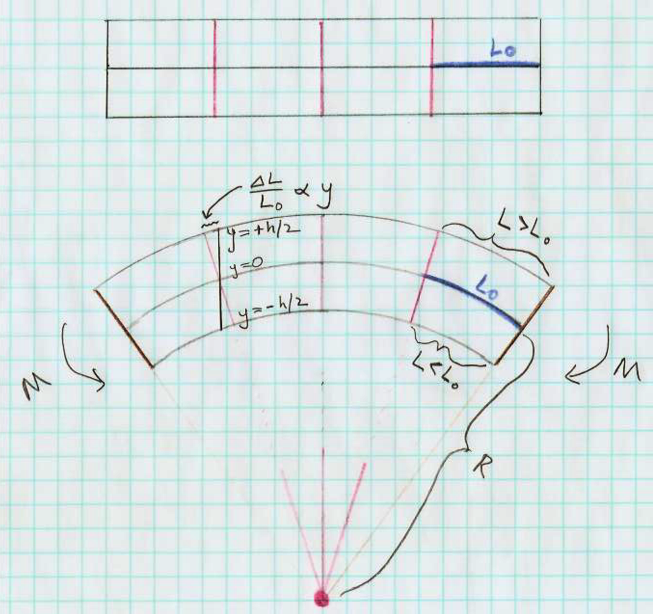
Navier's assumption: Sections that are originally plane and parallel remain plane after bending, but converge onto a common center of curvature. This assumption can be illustrated with a rubber beam.



Let's see how an initially horizontal beam responds to the bending moment $M(x)$ by deforming into a curved shape. In this illustration, top is in tension, as in a cantilever.



Key idea: bending moment $M \propto \frac{1}{R}$, where R is the radius of curvature of the beam. For constant M, vertical lines converge toward common center of curvature.



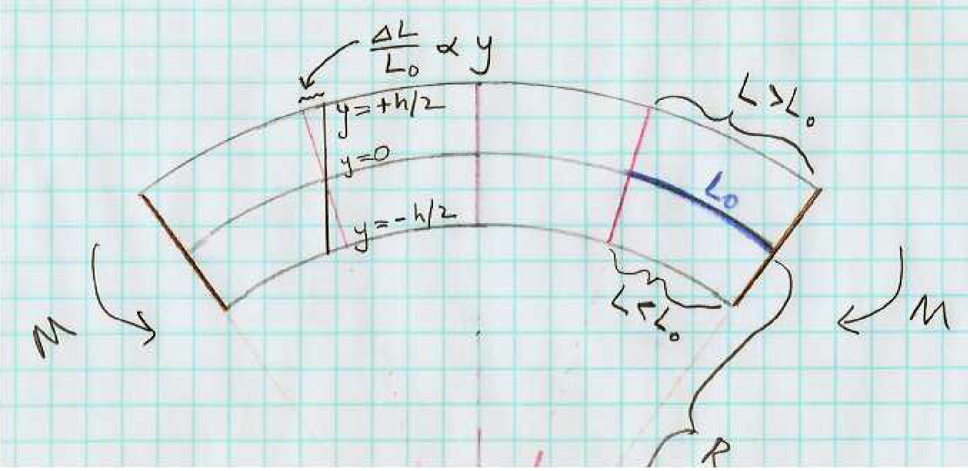
$$\text{strain} = \frac{\Delta L}{L_0} = \frac{y}{R}$$

where $y = 0$ is the neutral surface.

So in this case $y > 0$ is in tension and $y < 0$ is in compression.

If you think of wood fibers running along the beam's axis, then the fibers above the neutral surface ($y > 0$) are stretched in proportion to y, and the fibers below the neutral surface ($y < 0$) are compressed in proportion to $|y|$.

$$\text{strain} = \frac{\Delta L}{L} = \frac{y}{R}$$



Now remember that $\frac{\Delta L}{L}$ is called (axial) *strain*, and force per unit area is called *stress*. For an elastic material, strain (e) $\propto$ stress (f).

$$\frac{\Delta L}{L} = \frac{1}{E} \times \frac{\text{Force}}{\text{Area}} = \frac{1}{E} \times f \qquad e = \frac{1}{E} \times f$$

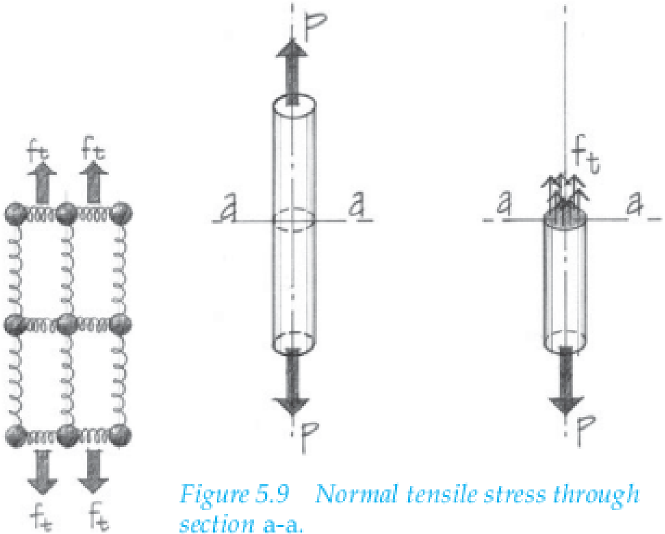
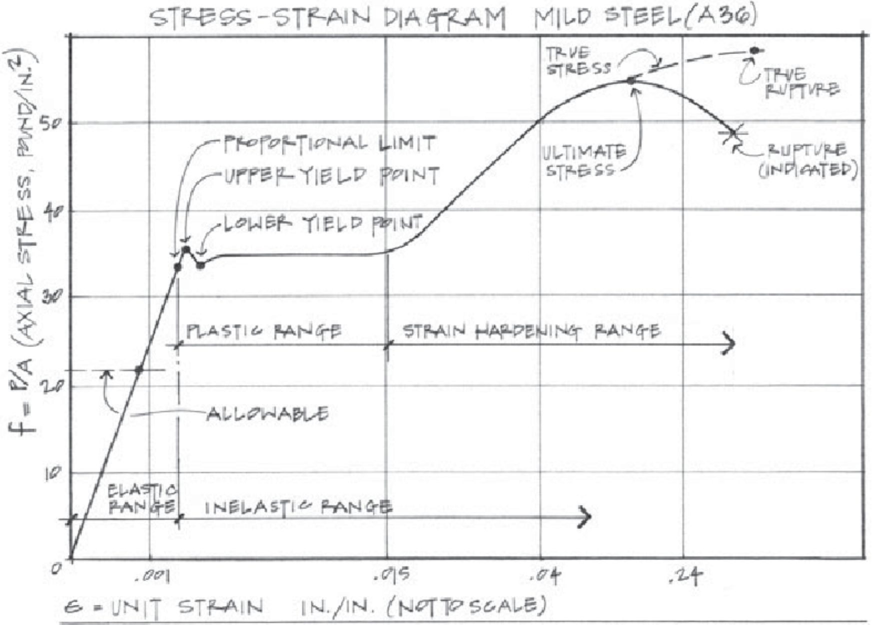


Figure 5.9 Normal tensile stress through section a-a.

In the elastic region, strain ($e = \Delta L/L$) is proportional to stress ($f = F/A$). $\boxed{f = Ee}$. The slope E is Young's modulus.

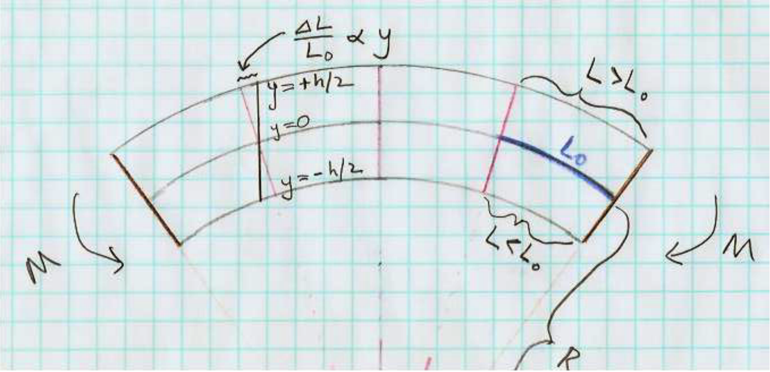


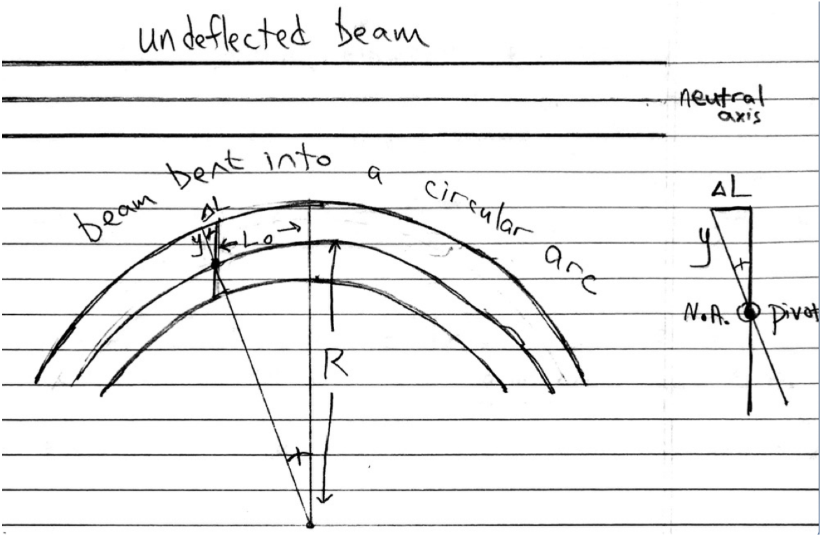
Plugging in $f = Ee$ to the bending-beam diagram:

$$\frac{y}{R} = \frac{\Delta L}{L} = e = \frac{f}{E}$$

we find the force-per-unit area (stress) exerted by the fibers is

$$f = \frac{Ey}{R}$$





By similar triangles, $\frac{\Delta L}{y} = \frac{L_0}{R} \quad \Rightarrow \quad$ strain $e = \frac{\Delta L}{L_0} = \frac{y}{R}$

Hooke's law: $f = eE \quad \Rightarrow \quad \boxed{f = \frac{Ey}{R}}$ (1)

f = stress = force per unit area. E = Young's modulus.

Imagine a fiber running along the length of the bent beam. Let the fiber have cross-section area $\mathrm{d}A$ and height y above the neutral surface. The tension (force) in the fiber is

$$\mathrm{d}F = f\,\mathrm{d}A = \frac{E}{R}\,y\,\mathrm{d}A$$

Pivoting about the neutral axis, the moment (torque) exerted by this fiber is (since y is the lever arm from the pivot)

$$\mathrm{d}M = y\,\mathrm{d}F = \frac{E}{R}\,y^2\,\mathrm{d}A$$

To find the total bending moment exerted by this cross-section of beam, we add up all of the fibers over the entire cross-section:

$$M = \frac{E}{R}\int y^2\,\mathrm{d}A = \frac{EI}{R} \qquad \text{where} \quad \boxed{I = \int y^2\,\mathrm{d}A} \quad (2)$$

- One factor of y comes from strain $\Delta L/L_0 \propto y$.
- The second factor of y is lever arm above the N.A.

So the beam's radius of curvature is $\boxed{R = \frac{EI}{M}}$ (3) (illustrate).

Combine (1) + (3) $\Rightarrow$ bending stress $f = \frac{E\,y}{E\,I/M} = \boxed{\frac{M\,y}{I} = f}$

The maximum bending stress is

$$f_{\max} = \frac{|M|_{\max}\,|y|_{\max}}{I} = \boxed{\frac{|M|_{\max}}{S} = f_{\max}}$$

where S is the "section modulus" $\boxed{S = \frac{I}{|y|_{\max}}}$

- know load & span $\rightarrow$ find $|M|_{\max}$
- know type of material $\rightarrow$ allowable $f_{\max}$

$$S_{\text{required}} \geq \frac{|M|_{\max}}{f_{\text{allowable}}}$$

tells you how "big" a beam cross-section you need for this load, span, & material, to meet the maximum-bending-stress criterion, which is a "strength" criterion (not a "stiffness" criterion).

In calculus, $\frac{1}{R}$ quantifies the "curvature" of a function $Y(x)$

$$\text{curvature} \; = \; \frac{1}{R} \; = \; \frac{Y''(x)}{[1+Y'(x)^2]^{3/2}} \; \approx \; Y''(x)$$

The curvature of a function is closely related to its second derivative $Y''(x)$. If the slope $|Y'(x)| \ll 1$, as is true for beams used in structures, then $\frac{1}{R} = Y''(x)$.

For clarity, I'll write $Y(x)$ for the shape of the deflected beam, and reserve y to denote height above the neutral surface.

$$Y''(x) \; = \; \frac{1}{R} \; = \; \frac{M(x)}{E\,I}$$

$$\text{slope} \;\; Y'(x) \; = \; \frac{1}{E\,I} \int M(x)\,\mathrm{d}x$$

$$Y(x) \; = \; \frac{1}{E\,I} \int \mathrm{d}x \int M(x)\,\mathrm{d}x$$

deflection under load $\Delta(x) = -Y(x)$. This is where you get $\Delta_{\max} = \frac{5wL^4}{384EI}$ for simply-supported beam with uniform load w, etc.

This Onouye/Kane figure writes "y" here for deflection, but I wrote "Y" for deflected beam shape, because we were already using y for "distance above the neutral surface."

You integrate $M(x)/(E\,I)$ twice w.r.t. x to get the deflected beam shape $Y(x)$.

The bending moment is $M(x) = E\,I\,\mathrm{d}^2Y/\mathrm{d}x^2$, where E is Young's modulus and I is second moment of area.

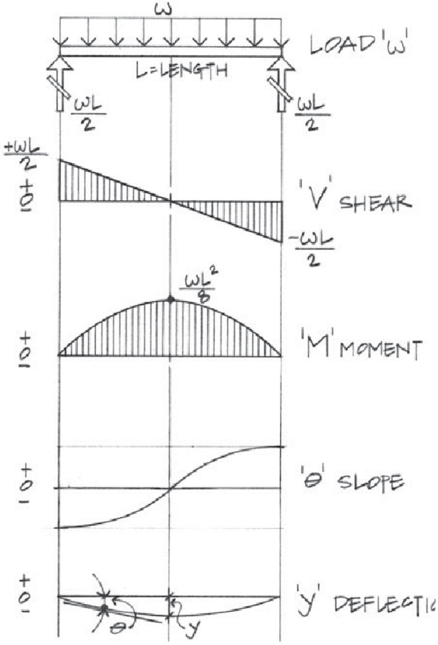
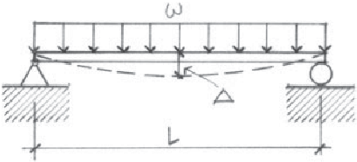
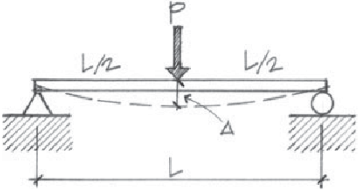
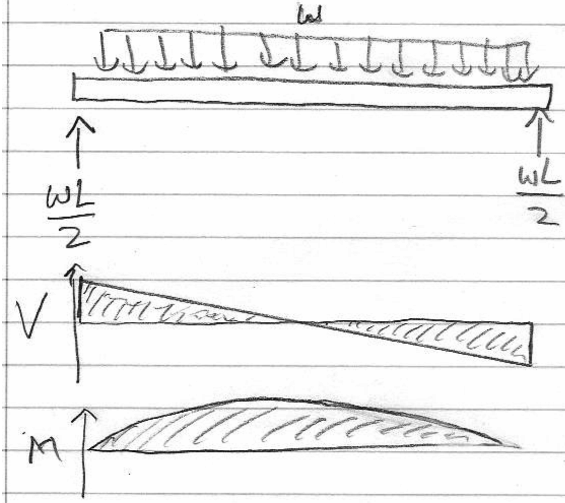


Figure 7.11 Relationship of load, shear, moment, slope, and deflection diagrams.

The most common deflection results can be found in tables.

Beam Load and Support	Actual Deflection*
 (a) Uniform load, simple span	$\Delta_{max} = \dfrac{5\omega L^4}{384EI}$ (at the centerline)
 (b) Concentrated load at midspan	$\Delta_{max} = \dfrac{PL^3}{48EI}$ (at the centerline)

FYI, here's where that crazy $(5wL^4)/(384EI)$ comes from!



$$V(x) = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$$

$$M(x) = \frac{wLx}{2} - \frac{wx^2}{2} = \frac{w}{2}\left(Lx - x^2\right)$$

(continued on next page)

Here's where that crazy $(5wL^4)/(384EI)$ comes from!

$$\Delta(x) = -\frac{1}{EI}\int dx \int M(x)\, dx$$

$$= -\frac{w}{2EI}\int dx \int (Lx - x^2)\, dx = -\frac{w}{2EI}\int dx \left(\frac{Lx^2}{2} - \frac{x^3}{3} + C_1\right)$$

$$= -\frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12} + C_1 x + C_2\right)$$

$$\Delta(0) = 0 \Rightarrow \boxed{C_2 = 0}$$

$$\Delta(L) = 0 \Rightarrow \frac{L^4}{6} - \frac{L^4}{12} + C_1 L = 0 \Rightarrow \boxed{C_1 = -\frac{L^3}{12}}$$

$$\Delta(x) = -\frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12} - \frac{L^3 x}{12}\right)$$

$$\Delta_{max} = \Delta\left(\frac{L}{2}\right) = -\frac{w}{2EI}\left(\frac{L(L/2)^3}{6} - \frac{(L/2)^4}{12} - \frac{L^3(L/2)}{12}\right)$$

$$= -\frac{wL^4}{2EI}\left(\frac{1}{48} - \frac{1}{192} - \frac{1}{24}\right) = -\frac{wL^4}{2EI}\left(-\frac{5}{192}\right) = \boxed{\frac{5wL^4}{384EI}}$$

The 2 integration constants can be tricky. Simply supported: $\Delta(0) = \Delta(L) = 0$. (For cantilever, $\Delta(0) = \Delta'(0) = 0$ instead.)

Maximum deflection is one of several beam-design criteria. $\Delta_{\max}$ comes from integrating $M(x)/(EI)$ twice w.r.t. x to get $\Delta(x)$.

For uniform load w on simply-supported beam, you get

$$\Delta_{\max} = \frac{5wL^4}{384EI}$$

You just look these results up, or use a computer to calculate them. But I had great fun calculating the 5/384 myself!

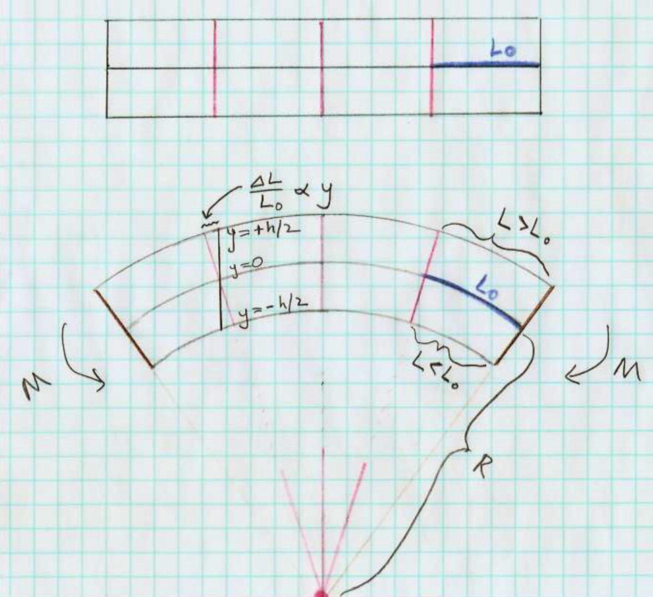
Deflection is proportional to load, and inversely proportional to Young's modulus and to the second moment of area.

- More load $\rightarrow$ more deflection
- Stiffer material $\rightarrow$ less deflection
- Cross-section with larger $I = \int y^2 \, \mathrm{d}A$ $\rightarrow$ less deflection

Notice that putting a column in the middle of a long, uniformly loaded beam reduces $\Delta_{\max}$ by a factor of $2^4 = 16$. Alternatively, if you want to span a large, open space without intermediate columns or bearing walls, you need beams with large I.

Recap: Bending beam into circular arc of radius R gives

$$\boxed{e = \frac{\Delta L}{L_0} = \frac{y}{R}}$$, strain e vs. distance y above the neutral surface.



Hooke's Law $\boxed{f = E\,e}$

gives stress $\boxed{f = \frac{E\,y}{R}}$

Torque exerted by fibers of beam is

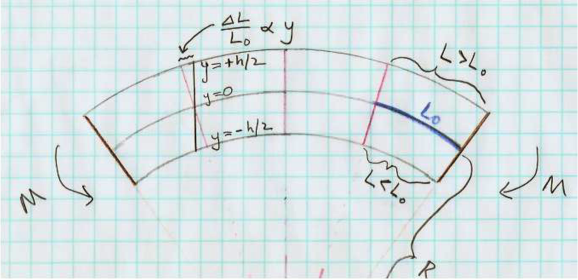
$$M = \int y\,(f\,\mathrm{d}A) =$$

$$y\,\frac{E\,y}{R}\,\mathrm{d}A = \frac{E}{R}\,y^2\,\mathrm{d}A$$

$$\boxed{M = \frac{E\,I}{R}}$$

Eliminate $R \Rightarrow$

$$\boxed{f = \frac{M\,y}{I} = \frac{M}{I/y}}$$



The bending stress a distance y above the neutral surface is

$$f = \frac{M\,y}{I}$$

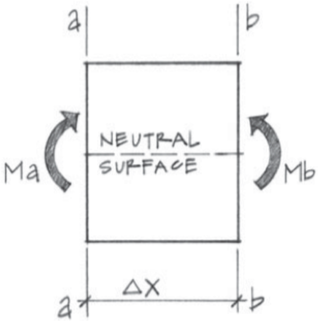
The largest bending stress happens in the fibers farthest above or below the neutral surface. Call this largest distance $y_{\max} \equiv c$.

$$f_{\max} \;=\; \frac{M_{\max}\,c}{I} \;=\; \frac{M_{\max}}{(I/c)} \;=\; \frac{M_{\max}}{S}$$

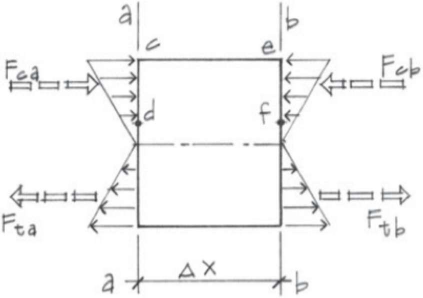
The ratio $S = I/c$ is called "section modulus." The load diagram gives you $M_{\max}$. Each material (wood, steel, etc.) has allowed bending stress $f_{\max}$. Then $S_{\min}$ tells you how big a beam you need.

Questions for Prof. Farley!

- How do we explain the variation of shear stress across the cross-section of a beam — for example: where is shear stress largest for a simply supported beam with uniform distributed load, rectangular cross-section?
- Should we add to this course some physics of masonry structures, e.g. a classic Roman arch?
- For design criteria of a structure (O/K ch1), what is meant by redundancy and continuity?
- Z.E. question: how to study moments in complex shapes?
- Any others?!



(a) Beam section between sections a *and* b.

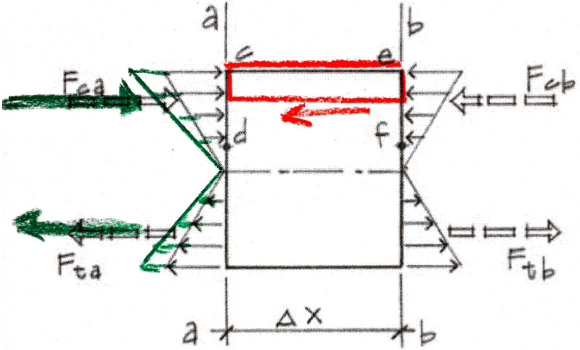


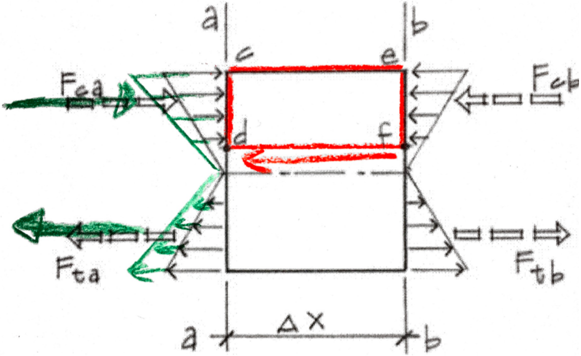
(b) Bending stresses on the beam section a-b.

Figure 8.19 Bending stress on a beam section.

Notice (on next slide):

- For most load/support conditions, bending moment $M(x)$ varies with x, and bending stress is proportional to bending moment.
- The shear stress (exerted between parallel fibers) along the bottom edge of the red rectangle must make up the difference between the left and right total bending forces.
- The left and right total bending forces depend on how much area we add up in drawing the red rectangle.
- The total reaches a maximum at the neutral surface, then decreases, since the direction of the bending stress reverses at the neutral surface.


a
b
c
e
F_ca
F_cb
d
f
F_ta
F_tb
ΔX
a
b


a
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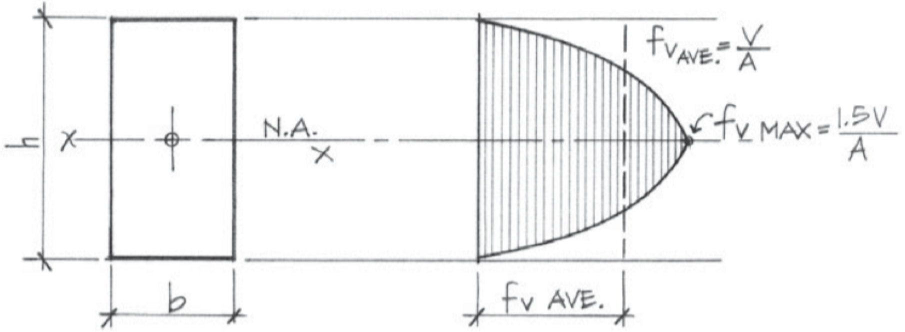
F_tb
ΔX
a
b

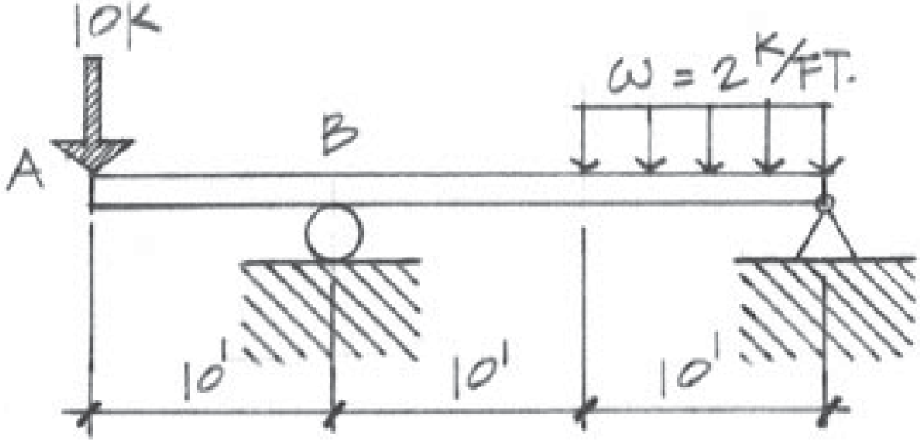


Cross Section.

Shear stress graph of a rectangular cross section.

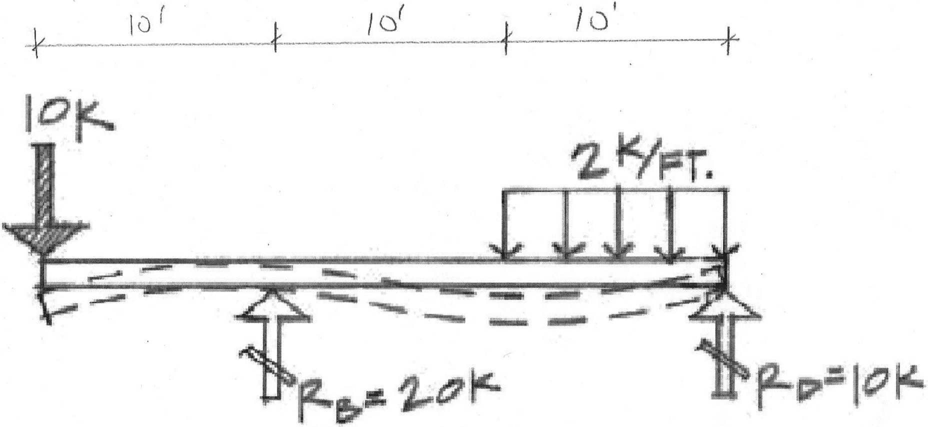
Figure 8.26 Shear stress distribution—key points.

If you find this confusing:
(a) You don't really need to know it for this course. If you're an architect, you'll learn it again when you study structures.
(b) You might look at the explanation I wrote up in the Onouye/Kane chapter 8 pages of
`http://positron.hep.upenn.edu/p8/files/equations.pdf#page=22`

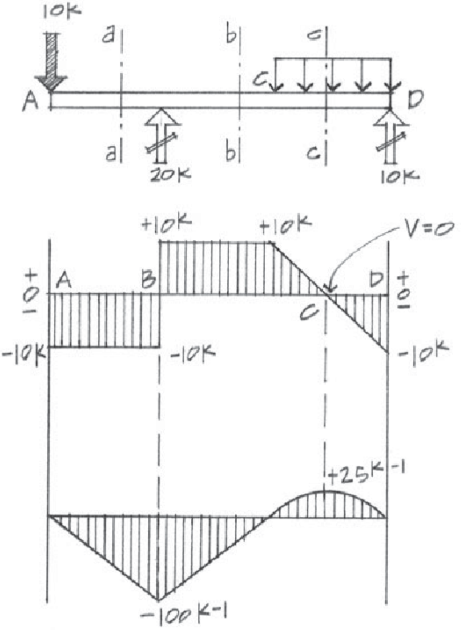


Draw shear (V) and moment (M) diagrams for this beam! Tricky! First one needs to solve for the support ("reaction") forces.

Note: in solving for the support forces, you replace distributed load w with equivalent point load. But when you draw the load diagram to find V and M, you need to keep w in its original form.



Remember that $V(x)$ is the running sum, from LHS to x, of vertical forces acting on the beam, with upward=positive.



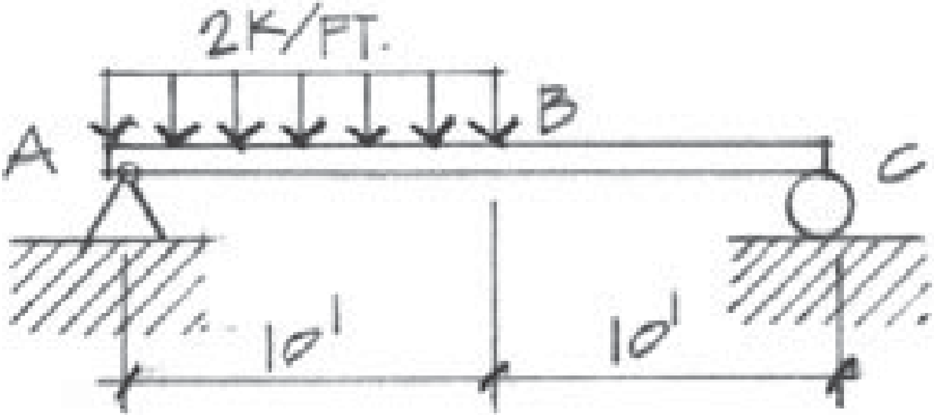
Load diagram.

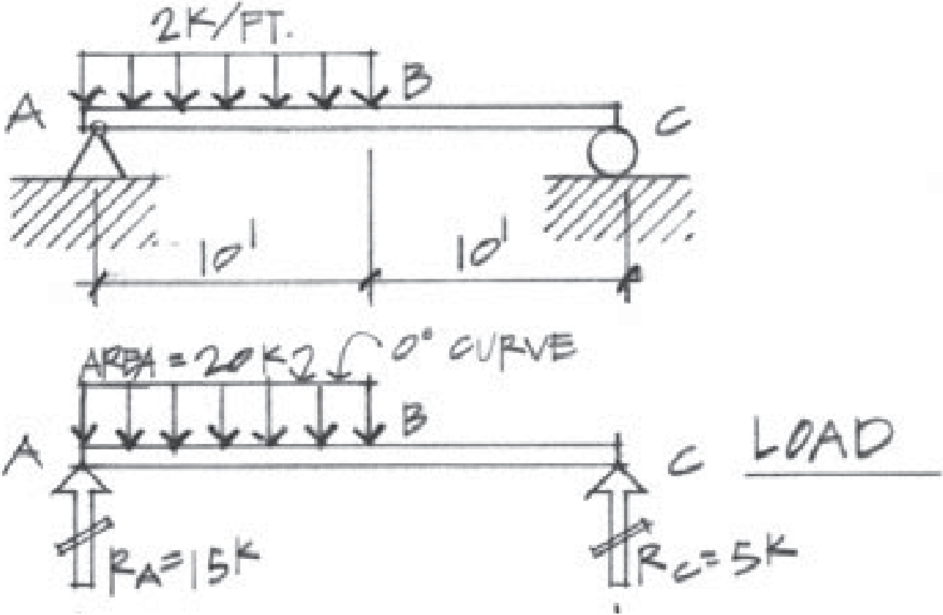
Shear diagram.

Moment diagram.

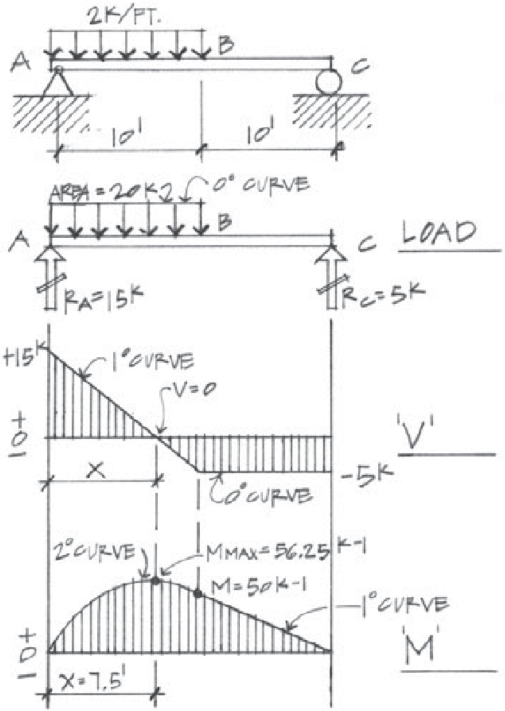
Neat trick: $M_2 - M_1 = (V_{1\to 2}^{\text{average}})(x_2 - x_1)$

Draw load, V, and M diagrams for this simply supported beam with a partial uniform load.





2K/FT.
A
B
C
10'
10'
AREA = 20K
2
0° CURVE
A
B
C
LOAD
RA=15K
RC=5K


2K/FT.
A
B
C
10'
10'
AREA = 20K
0° CURVE
LOAD
RA=15K
RC=5K
+15K
1° CURVE
V=0
+
0
−
'V'
X
-5K
0° CURVE
2° CURVE
MMAX=56.25K-1
M=50K-1
1° CURVE
'M'
X=7.5'

(The next few slides contain beam-design examples, of the sort you might see in a structures course.)

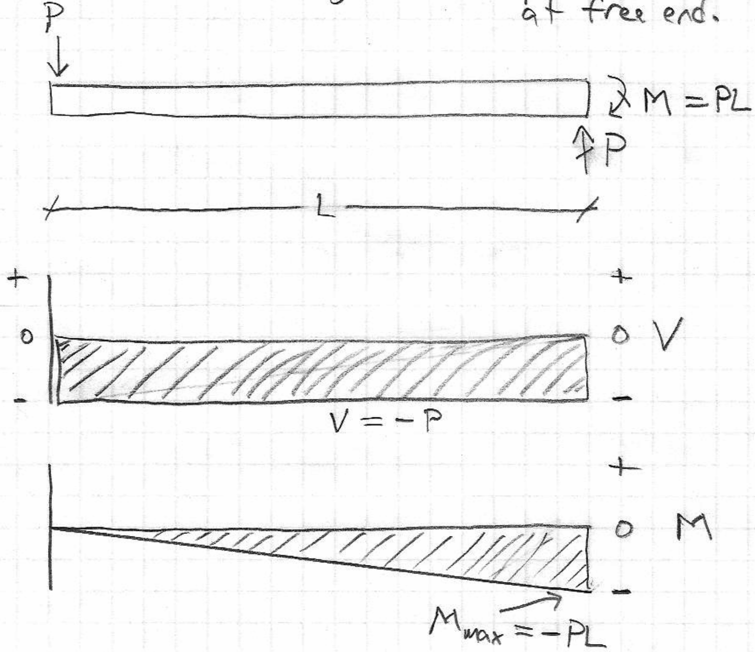
Example (using metric units!): A cantilever beam has a span of 3.0 m with a single concentrated load of 100 kg at its unsupported end. If the beam is made of timber having allowable bending stress $F_b = 1.1 \times 10^7\ \mathrm{N/m^2}$ (was 1600 psi in US units), what minimum section modulus is required?

What is the smallest "2×" dimensional lumber (width = 1.5 inch = 0.038 m) whose cross-section satisfies this strength criterion?

Would this beam also satisfy a $\Delta_{\mathrm{max}} < L/240$ (maximum deflection) stiffness criterion? If not, what standard "2×" cross-section is needed instead?

$\Delta_{\mathrm{max}} = PL^3/(3EI)$ for a cantilever with concentrated load at end. Use Young's modulus $E = 1.1 \times 10^{10}\ \mathrm{N/m^2}$ for southern pine.

Cantilever of length L with point load P at free end.



$$S_{\min} = \frac{|M_{\max}|}{f_{\text{allowed}}} = \frac{PL}{F_b} = \frac{(980\,\mathrm{N})(3\,\mathrm{m})}{1.1 \times 10^7\,\mathrm{N/m^2}} = 26.7 \times 10^{-5}\,\mathrm{m}^3$$

$$\Delta_{\text{allowed}} = \frac{L}{240} = \frac{3.0\,\mathrm{m}}{240} = 0.0125\ \mathrm{m}$$

$$\Delta_{\max} = \frac{PL^3}{3EI} \quad \Rightarrow \quad I_{\min} = \frac{PL^3}{3E\Delta_{\text{allowed}}} = 64.2 \times 10^{-6}\,\mathrm{m}^4$$

I worked out b, h, I, and $S = I/c$ values in metric units for standard "2×" dimensional lumber.

	b	b	h	h	$I = bh^3/12$	$S = bh^2/6$
2×4	1.5 in	.038 m	3.5 in	.089 m	$2.23 \times 10^{-6}\,\mathrm{m}^4$	$5.02 \times 10^{-5}\,\mathrm{m}^3$
2×6	1.5 in	.038 m	5.5 in	.140 m	$8.66 \times 10^{-6}\,\mathrm{m}^4$	$12.4 \times 10^{-5}\,\mathrm{m}^3$
2×8	1.5 in	.038 m	7.5 in	.191 m	$21.9 \times 10^{-6}\,\mathrm{m}^4$	$23.0 \times 10^{-5}\,\mathrm{m}^3$
2×10	1.5 in	.038 m	9.5 in	.241 m	$44.6 \times 10^{-6}\,\mathrm{m}^4$	$37.0 \times 10^{-5}\,\mathrm{m}^3$
2×12	1.5 in	.038 m	11.5 in	.292 m	$79.1 \times 10^{-6}\,\mathrm{m}^4$	$54.2 \times 10^{-5}\,\mathrm{m}^3$

The numbers are nicer if you use centimeters instead of meters, but then you have the added hassle of remembering to convert back to meters in calculations.

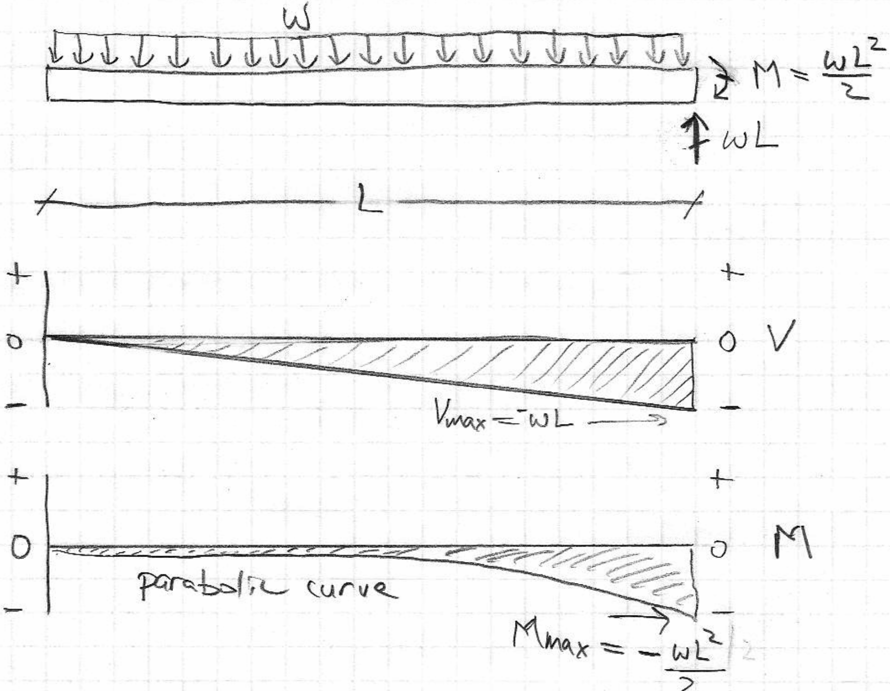
	b	b	h	h	$I = bh^3/12$	$S = bh^2/6$
2×4	1.5 in	3.8 cm	3.5 in	8.9 cm	$223\,\mathrm{cm}^4$	$50.2\,\mathrm{cm}^3$
2×6	1.5 in	3.8 cm	5.5 in	14.0 cm	$866\,\mathrm{cm}^4$	$124\,\mathrm{cm}^3$
2×8	1.5 in	3.8 cm	7.5 in	19.1 cm	$2195\,\mathrm{cm}^4$	$230\,\mathrm{cm}^3$
2×10	1.5 in	3.8 cm	9.5 in	24.1 cm	$4461\,\mathrm{cm}^4$	$370\,\mathrm{cm}^3$
2×12	1.5 in	3.8 cm	11.5 in	29.2 cm	$7913\,\mathrm{cm}^4$	$542\,\mathrm{cm}^3$

Minor variation on same problem: A cantilever beam has a span of $3.0\ \mathrm{m}$ with a uniform distributed load of $33.3\ \mathrm{kg/m}$ along its entire length. If we use timber with allowable bending stress $F_b = 1.1 \times 10^7\ \mathrm{N/m^2}$, what minimum section modulus is required?

What is the smallest "$2\times$" dimensional lumber (width $= 1.5$ inch $= 0.038$ m) whose cross-section satisfies this strength criterion?

Would this beam also satisfy a $\Delta_{\mathrm{max}} < L/240$ (maximum deflection) stiffness criterion? If not, what standard "$2\times$" cross-section is needed instead?

$\Delta_{\mathrm{max}} = wL^4/(8EI)$ for a cantilever with uniform load. Use Young's modulus $E = 1.1 \times 10^{10}\ \mathrm{N/m^2}$ for southern pine.


w
M = wL²/2
wL
L
+
0
−
+
0
−
V
Vmax = −wL
+
0
−
+
0
−
M
parabolic curve
Mmax = −wL²/2

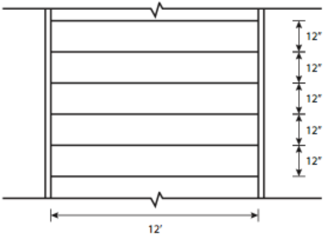
$$S_{\min} = \frac{|M_{\max}|}{f_{\text{allowed}}} = \frac{wL^2/2}{F_b} = \frac{(326\,\mathrm{N/m})(3\,\mathrm{m})^2/2}{1.1\times 10^7\,\mathrm{N/m^2}} = 13.3\times 10^{-5}\,\mathrm{m}^3$$

$$\Delta_{\text{allowed}} = \frac{L}{240} = \frac{3.0\,\mathrm{m}}{240} = 0.0125\ \mathrm{m}$$

$$\Delta_{\max} = \frac{wL^4}{8EI} \quad\Rightarrow\quad I_{\min} = \frac{wL^4}{8E\Delta_{\text{allowed}}} = 24.0\times 10^{-6}\,\mathrm{m}^4$$

2) Size a wood joist for a row house floor which spans 12 feet. Joists are spaced at 16 inches on center.

f = 1,300 psi
f = 85 psi
E = 1.7 x 10 psi
LL = 60 psf
DL =30 psf



Plan View

Hint: remember that a "2 x 4" wood joist is only nominal; its true dimensions are "1.5 x 3.5" inches.
(4 = 1.5, 6 = 5.5, 8 = 7.25, 10 = 9.25 inches)

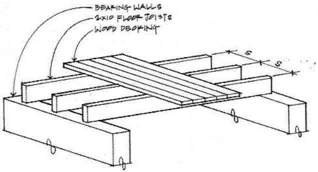
(Here's a homework problem from ARCH 435.)

Actually, Home Depot's 2×10 really is 9.5 inches deep, not 9.25 inches, and 2×12 really is 11.5 inches deep.

A timber floor system uses joists made of "2×10" dimensional lumber. Each joist spans a length of 4.27 m (simply supported). The floor carries a load of 2400 $\mathrm{N/m^2}$. At what spacing should the joists be placed, in order not to exceed allowable bending stress $F_b = 10000\ \mathrm{kN/m^2}$ $(1.0 \times 10^7\ \mathrm{N/m^2})$?

(We should get an answer around 24 inches = 0.61 meters.)

8.6 A timber floor system utilizing 2 × 10 S4S joists spans a length of 14′ (simply supported). The floor carries a load of 50 psf (DL + LL). At what spacing should the joists be placed? Assume Douglas Fir–Larch No. 2 (F_b = 1,450 psi).



Solution:

Based on the allowable stress criteria:

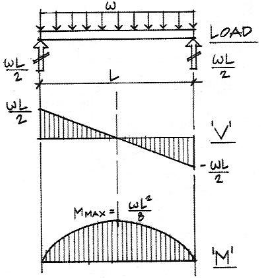
$$f = \frac{Mc}{I} = \frac{M}{S}$$

$$M_{max} = S \times f_b = (21.4 \text{ in.}^3)(1.45 \text{ k/in.}^3) = 31 \text{ k-in.}$$

$$M = \frac{31 \text{ k-in.}}{12 \text{ in./ft.}} = 2.58 \text{ k-ft.}$$

Based on the bending moment diagram:

$$M_{max} = \frac{\omega L^2}{8}$$



Therefore,

$$\omega = \frac{8M}{L^2}$$

Substituting for M obtained previously,

$$\omega = \frac{8(2.58 \text{ k-ft.})}{(14')^2} = 0.105 \text{ k/ft.} = 105 \text{ \#/ft.}$$

But

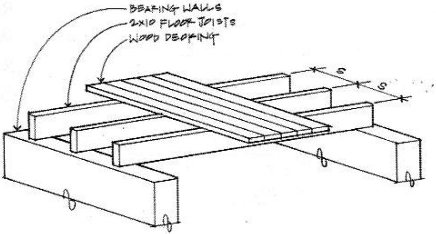
$$\omega = \text{\#/ft.}^2 \times \text{tributary width (joist spacing } s)$$

$$s = \frac{\omega}{50 \text{ psf}} = \frac{105 \text{ \#/ft.}}{50 \text{ \#/ft.}^2} = 2.1'$$

$$s = 25'' \text{ spacing}$$

Use 24″ o.c. spacing.

8.6 A timber floor system utilizing 2 × 10 S4S joists spans a length of 14′ (simply supported). The floor carries a load of 50 psf (DL + LL). At what spacing should the joists be placed? Assume Douglas Fir–Larch No. 2 (F_b = 1,450 psi).



Solution:

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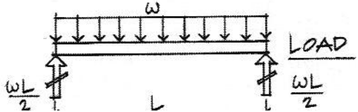

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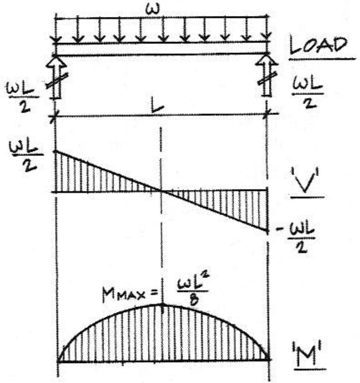

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$$s = 25'' \text{ spacing}$$

Use 24" o.c. spacing.

Note: *Spacing is more practical for plywood subflooring, based on a 4 ft. module of the sheet.*