Derivation of the "mass law" of sound attenuation at a single, uniform wall or window.

The problem we're really interested in is what happens to a sound wave when it encounters a single window or wall of uniform mass per unit area. In class, we modeled this problem by putting a mass onto the middle of the wave machine. The result was that a small part of the wave was transmitted to the other side of the mass, while most of the wave was reflected back by the mass. The transmission was smaller if we increased the mass, and it was smaller for larger frequencies. Low frequencies were very difficult to stop.

Here, I'll do the math for the analogous case of putting a mass onto the middle of a taut string (like a guitar string), and calculating how much of an incoming wave is transmitted and how much is reflected back.

I'll put the mass at $x = 0$. The incoming wave comes from $x < 0$ and is traveling to the right. For $x > 0$, we have only the transmitted wave, which travels to the right. For $x < 0$ we have both the incoming wave (which travels to the right) and the reflected wave (which travels to the left).

Remember that a single-frequency wave traveling to the right is described by the wavefunction

$$\psi(x,t) = A\cos(\omega t - kx + \phi)$$

and a single-frequency wave traveling to the left is described by the wavefunction

$$\psi(x,t) = A\cos(\omega t + kx + \phi)$$

For the incoming wave, we can choose the amplitude to be $A = 1$ (so that the amplitudes we report are just fractions of the incoming wave's amplitude) and we can choose the phase to be $\phi = 0$ (which just fixes the time at which we start the clock). For the reflected and transmitted waves, the amplitude and phase will be determined by the physics at the boundary. So we can write the wavefunctions for the left ($x < 0$) and right ($x > 0$) sides of the mass like this:

$$\psi_L(x,t) = \cos(\omega t - kx) + R\cos(\omega t + kx + \phi_R)$$

$$\psi_R(x,t) = T\cos(\omega t - kx + \phi_T)$$

But we will be stuck with a lot of messy trigonometry if we write the phases ϕ_R and ϕ_T of the reflected and transmitted waves in that way. The math is much more reasonable if instead of using a cosine function with a phase, we instead allow for a linear combination of cosine and sine functions. This has exactly the same effect as allowing for an arbitrary phase ϕ. So we'll write the wavefunctions like this:

$$\psi_L(x,t) = \cos(\omega t - kx) + R_1\cos(\omega t + kx) + R_2\sin(\omega t + kx)$$

$$\psi_R(x,t) = T_1 \cos(\omega t - kx) + T_2 \sin(\omega t - kx)$$

Now we need to work out the boundary conditions at the mass. The first boundary condition is that the string is continuous at $x = 0$. We assume that the mass is very thin, so the height of the string just to the left of $x = 0$ must be the same as the height of the string just to the right of $x = 0$. In math, this condition is

$$\psi_L(0,t) = \psi_R(0,t)$$

Plugging this in, we get

$$\cos(\omega t) + R_1 \cos(\omega t) + R_2 \sin(\omega t) = T_1 \cos(\omega t) + T_2 \sin(\omega t)$$

This equation must be true at all possible values of ωt. So we evaluate it first for $\omega t = 0$, when $\cos(\omega t) = 1$ and $\sin(\omega t) = 0$, and then for $\omega t = \frac{\pi}{2}$, when $\cos(\omega t) = 0$ and $\sin(\omega t) = 1$. This gives us two equations:

$$1 + R_1 = T_1 \quad \text{and} \quad R_2 = T_2$$

Substituting these two equations into ψ_L lets us eliminate R_1 and R_2, so we can write

$$\psi_L(x,t) = \cos(\omega t - kx) + (T_1 - 1)\cos(\omega t + kx) + T_2 \sin(\omega t + kx)$$

$$\psi_R(x,t) = T_1 \cos(\omega t - kx) + T_2 \sin(\omega t - kx)$$

The second boundary condition is basically $F = ma$ for the mass. As the mass wiggles up and down, it is being accelerated by the two sides of the string. If the string is perfectly horizontal, then the tension in the string is perfectly horizontal, and the mass is not accelerated. (We're only considering vertical motion of the string—as we saw in class on the transverse wave machine.) If the string has a slope, then the tension has a vertical component. If the left and right sides of the string have different slopes, then the mass feels a net force in proportion to this difference in slopes. Writing $F = ma$ for the mass in terms of the string tension τ, we have

$$\left(\psi_R'(0,t) - \psi_L'(0,t)\right)\tau = m\ddot{\psi}(0,t)$$

For the acceleration, I wrote $\ddot{\psi}(0,t)$ without distinguishing left or right, because we know $\psi_L(0,t) = \psi_R(0,t)$, so their time-derivatives must also be equal. The second derivative of the wavefunction (we'll use ψ_R, but for $x = 0$ we could choose either ψ_R or ψ_L) with respect to time is

$$\ddot{\psi}_R(x,t) = -\omega^2 T_1 \cos(\omega t - kx) - \omega^2 T_2 \sin(\omega t - kx)$$

The slopes of the left and right sides of the string are

$$\psi_R'(x,t) = kT_1 \sin(\omega t - kx) - kT_2 \cos(\omega t - kx)$$

$$\psi'_L(x,t) = k\sin(\omega t - kx) + k(T_1 - 1)\sin(\omega t + kx) + kT_2\cos(\omega t + kx)$$

Plugging these in for $x = 0$, we get

$$(kT_1\sin(\omega t) - kT_2\cos(\omega t)) - (k\sin(\omega t) + k(T_1 - 1)\sin(\omega t) + kT_2\cos(\omega t))$$
$$= (m/\tau)\left(-\omega^2 T_1\cos(\omega t) - \omega^2 T_2\sin(\omega t)\right)$$

Making this be true for all ωt gives two equations:

$$kT_1 - k - k(T_1 - 1) = \left(-\omega^2 m/\tau\right) T_2$$

$$-kT_2 - kT_2 = \left(-\omega^2 m/\tau\right) T_1$$

which simplify to

$$1 - T_1 = \frac{\omega^2 m}{k\tau} T_2 \quad \text{and} \quad T_2 = \frac{\omega^2 m}{2k\tau} T_1$$

or better yet (defining $\alpha = \frac{\omega^2 m}{2k\tau}$)

$$1 - T_1 = 2\alpha T_2 \quad \text{and} \quad T_2 = \alpha T_1$$

We can solve these two equations in two unknowns to get

$$T_1 = \frac{1}{1 + \alpha^2} \quad \text{and} \quad T_2 = \frac{\alpha}{1 + \alpha^2}$$

We get the transmitted amplitude (as a fraction of the incoming amplitude) by combining the cosine-like and sine-like components, like this:

$$T_{\text{amplitude}} = \sqrt{T_1^2 + T_2^2} = \frac{1}{\sqrt{1 + \alpha^2}}$$

The fraction of intensity transmitted (i.e. the intensity for $x > 0$ divided by the intensity of the incoming wave) is the square of the fraction of transmitted amplitude:

$$T_{\text{intensity}} = \frac{1}{1 + \alpha^2}$$

Now we can put our expression for α into a more useful form:

$$\alpha = \frac{\omega^2 m}{k\tau} = \frac{(2\pi f)^2 m}{(2\pi f/c)(\rho c^2)} = \frac{\pi m f}{c\rho}$$

using $\omega = 2\pi f$, $k = 2\pi f/c$, $c = \sqrt{\tau/\rho}$ (so $\tau = \rho c^2$), where ρ is the mass per unit length of the string, f is the frequency, and c is the speed of wave propagation on the string. So for transmitted intensity, we get

$$T_{\text{intensity}} = \frac{1}{1 + \left(\frac{\pi m f}{c\rho}\right)^2}$$

For all but the lowest frequencies, $\alpha \gg 1$, so

$$T_{\text{intensity}} \approx \left(\frac{c\rho}{\pi m f}\right)^2$$

To apply this to sound waves hitting a wall or window, we replace c with the speed of sound, ρ with the density of air (mass per unit volume), and m with the mass per unit area of the wall or window. I'll use the symbol μ instead of m to remind us that it is mass per unit area. We get

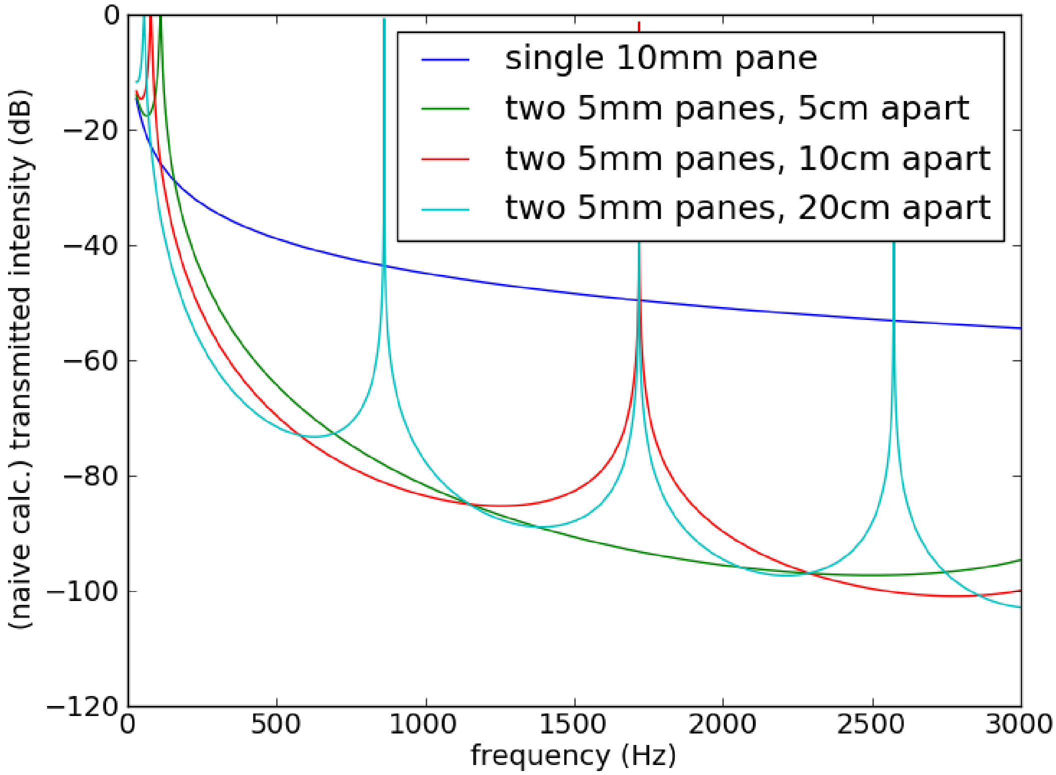
$$T_{\text{intensity}} = \frac{1}{1 + \left(\frac{\pi\mu f}{c\rho}\right)^2} \approx \left(\frac{c_{\text{sound}}\ \rho_{\text{air}}}{\pi\ \mu_{\text{wall}}\ f}\right)^2$$

This is sometimes called the "mass law" of sound reduction. (A 6 dB drop in sound level per doubling of mass or per doubling of frequency.) It applies for a single wall of uniform mass per unit area. By treating the wall as a single mass, I am basically assuming that the wall or window is perfectly rigid (doesn't change shape) and moves back and forth in response to the sound wave, like the mass we hung on the wave machine. A more complicated derivation would treat the wall like a short segment of much heavier string—like taking a thin guitar wire and splicing a small section of thick guitar wire into the middle. The two derivations give the exact same result as long as the wavelength of sound (in the wall/window material) is much longer than the thickness of the wall/window. The speed of sound in most building materials is about 10× as large as in air, so the wavelengths are larger by the same factor, which makes it a reasonable approximation for ordinary sound frequencies that the thickness of a window or a wall is much smaller than a wavelength.

You get an enormously larger reduction (except at resonant frequencies, where an odd number of half-wavelengths just fit between the two windows), with a given amount of material, by making two separated (and mechanically decoupled) walls or windows. The larger the separation, the better the attenuation for the lowest-frequency sounds. You can derive the sound transmission for a double window using similar (but much more messy) math. That result, in case you're curious, is

$$T_{\text{intensity}} = \frac{1}{1 + 4\alpha^2(C^2 - 2\alpha SC + \alpha^2 S^2)}$$

where $C = \cos(2\pi\Delta x/\lambda) = \cos(2\pi f \Delta x/c_{\text{sound}})$, $S = \sin(2\pi\Delta x/\lambda) = \sin(2\pi f \Delta x/c_{\text{sound}})$, Δx is the spacing between the two windows or walls, and λ is the wavelength in air. There is a graph of this function in the class notes for January 23.



single 10mm pane
two 5mm panes, 5cm apart
two 5mm panes, 10cm apart
two 5mm panes, 20cm apart
(naive calc.) transmitted intensity (dB)
0
−20
−40
−60
−80
−100
−120
0
500
1000
1500
2000
2500
3000
frequency (Hz)