Physics 351, Spring 2015, Homework #1.

Due at start of class, Friday, January 23, 2015

Instead of killing trees to hand out a printed syllabus, I'll just point you to the course web site, which contains a detailed syllabus that I will keep up-to-date throughout the term:

`positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at

`positron.hep.upenn.edu/q351`

to tell me how the homework went for you.

1. Let $\mathbf{u}$ be an arbitrary fixed unit vector. Show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2.$$

Explain this result in words, with the help of a picture.

2. If $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}).$$

Hint: Note that the derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$ distributes over vector products (dot product, cross product) analogously to the way it does over ordinary products. So for example,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a} \cdot \mathbf{b}) = \dot{\mathbf{a}} \cdot \mathbf{b} + \mathbf{a} \cdot \dot{\mathbf{b}}.$$

3. The two vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the xy plane and make angles α and β with the x axis. (a) By evaluating the dot product $\mathbf{a} \cdot \mathbf{b}$ in two ways [namely using equations (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta.$$

(b) By similarly evaluating $\mathbf{a} \times \mathbf{b}$ prove that

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta.$$

4. Prove that the magnetic forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ between two steady current loops (for which there is no electromagnetic wave to carry away momentum) obey Newton's third law. Hints: Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $\mathrm{d}\mathbf{r}_1$ and $\mathrm{d}\mathbf{r}_2$ are short segments of the loops, then according to the Biot-Savart law, the force on $\mathrm{d}\mathbf{r}_1$ due to $\mathrm{d}\mathbf{r}_2$ is

$$\frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2}\,\mathrm{d}\mathbf{r}_1 \times (\mathrm{d}\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$, and $\hat{\mathbf{s}} = \mathbf{s}/s$. The force $\mathbf{F}_{12}$ is found by integrating around both loops. Start by writing down the force on $\mathrm{d}\mathbf{r}_1$ due to $\mathrm{d}\mathbf{r}_2$, and expand it using the "BAC-CAB" rule. Do the same thing for the force on $\mathrm{d}\mathbf{r}_2$ due to $\mathrm{d}\mathbf{r}_1$. Each force will have two terms. One term in each force will involve $\mathrm{d}\mathbf{r}_1 \cdot \mathrm{d}\mathbf{r}_2$, and you can show that they are the negative of each other. You should be able to show that the other term in each force is of the form $\oint \nabla f \cdot \mathrm{d}\mathbf{r} = \oint \mathrm{d}f = 0$, i.e. the line integral of the gradient of a scalar function is zero around a closed path. This argument thus establishes that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

5. The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following experiment: I am standing on the ground (which we take to be an inertial frame, called frame $\mathcal{S}$) beside a perfectly flat horizontal turntable, rotating with constant angular velocity ω. I lean over and shove a frictionless puck so that it slides across the turntable, straight through the center. The puck is subject to zero net force and, as seen from my inertial frame, travels in a straight line. (a) Write down the polar coordinates r, ϕ of the puck as functions of time, as measured in the inertial frame $\mathcal{S}$ of the observer on the ground. (Assume that the puck was launched along the axis $\phi = 0$ at $t = 0$.) (b) Now write down the polar coordinates r', ϕ' of the puck as measured by an observer (frame $\mathcal{S}'$) at rest on the turntable. (Choose these coordinates so that ϕ and ϕ' coincide at $t = 0$.) (c) Describe and sketch the path as seen by this second observer. Is the frame $\mathcal{S}'$ inertial?

6. The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. (a) Use Mathematica (or other software if you prefer) to solve the differential equation for the case that the board is released from $\phi_0 = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of $\phi(t)$ for two or three periods. (b) On the same picture, plot the approximate solution (1.57) with the same $\phi_0 = 20°$. Comment on your two graphs. (c) Repeat parts (a) and (b) using the initial value $\phi_0 = \pi/2$ and comment. You will need to learn to use Mathematica's `NDSolve` command and to plot the solution that it provides using the `Plot` command. The `Plot` command can also graph the approximate solution (1.57). The graph is most informative if you overlay the numerical solution and the approximate solution on the same axes for direct comparison. I'll illustrate in class how to do these things in Mathematica.

7. There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in Chapter 2) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. (a) Write down Newton's second law and separate the variables by rewriting it as $m\frac{\mathrm{d}v}{F(v)} = \mathrm{d}t$. Now integrate both sides of this equation and show that

$$t = m\int_{v_0}^{v} \frac{\mathrm{d}v'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. (b) Use this method to solve the special case that $F(v) = F_0$, a constant force, and comment on your result. (c) Next, use the same method to solve for the case in which a mass m has velocity v_0 at time $t = 0$ and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Find v in terms of the time t and the other given parameters. At what time (if any) will the mass come to rest?

8. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\mathrm{d}x'.$$

Hint: use the chain rule to prove the following handy relation: If you regard v as a function of x, then

$$\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}x}.$$

Use the above relation to rewrite Newton's second law in the separated form $m\ \mathrm{d}(v^2) = 2F(x)\ \mathrm{d}x$ and then integrate from x_0 to x. Comment on your result in the case that $F(x)$ is actually a constant. (You should recognize your solution as a statement about kinetic energy and work.)

9. Use the method of Problem 7 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_0 e^{v/V}$, where F_0 and V are constants. (a) Find $v(t)$ if the initial velocity is $v_0 > 0$ at time

$t = 0$. (b) At what time does the mass come instantaneously to rest? (c) By integrating $v(t)$, you can find $x(t)$. Do this and find how far the mass travels before coming instantaneously to rest.

10. A basketball has mass $m = 600$ g and diameter $D = 24$ cm. (a) What is its terminal speed in air? (b) If it is dropped from a 30 m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

11. Consider the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$. (a) By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. (b) Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

12. A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, $\mathbf{E}$ pointing in the y direction and $\mathbf{B}$ in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{x0}$ (positive or negative). (a) Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$. (b) Prove that there is a unique value of v_{x0}, called the drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam with many different speeds.) Note that "undeflected" means that the velocity (vector) is constant. (c) Solve the equations of motion to give the particle's velocity as a function of t, for arbitrary values of v_{x0}. [Hint: The equations for (v_x, v_y) should look very much like Equations (2.68) except for an offset of v_x by a constant. If you make a change of variables of the form $u_x = v_x - v_{\mathrm{dr}}$ and $u_y = v_y$, the equations for (u_x, u_y) will have exactly the form (2.68), whose general solution you know.] (d) Integrate the velocity to find the position as a function of t and sketch the trajectory for various values of v_{x0}. As a check on your answer, you should find for part (d) that $x = v_{\mathrm{dr}}t + R\sin\omega t$ and $y = R(\cos\omega t - 1)$, where $R = (v_{x0} - v_{\mathrm{dr}})/\omega$ and $\omega = qB/m$.

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(extra credit on following pages)

XC1*. Optional/extra-credit. Suppose that the basketball of Problem 10 is thrown from a height of 2 m with initial velocity $\mathbf{v}_0 = 15$ m/s at 45° above the horizontal. (a) Use Mathematica (or some other system that you already know) to solve the equations of motion (2.61) for the ball's position (x, y) and plot the trajectory. Also plot the corresponding trajectory in the absence of air resistance. (b) Use your plot to find how far the ball travels in the horizontal direction before it hits the floor. Compare with the corresponding range in a vacuum.

XC2*. Optional/extra-credit. The equation (2.39) for the range of a projectile in a linear medium cannot be solved analytically in terms of elementary functions. If you put in numbers for the several parameters, then it *can* be solved numerically using Mathematica (or similar). To practice this, do the following: Consider a projectile launched at angle θ above the horizontal ground with initial speed v_0 in a linear medium. Choose units such that $v_0 = 1$ and $g = 1$. Suppose also that the terminal speed $v_{\text{ter}} = 1$. (With $v_0 = v_{\text{ter}}$, air resistance should be fairly important.) We know that in a vacuum, the maximum range occurs at $\theta = \pi/4 \approx 0.75$. (a) What is the maximum range in a vacuum? (b) Now solve (2.39) for the range in the given medium at the same angle $\theta = 0.75$. (c) Once you have your calculation working, repeat it for some selection of values of θ within which the maximum range probably lies — e.g. you could try $\theta = 0.4, 0.5, \cdots, 0.8$. (d) Based on these results, choose a smaller interval for θ where you're sure the maximum lies and repeat the process. Repeat it again if necessary until you know the maximum range and the corresponding angle to two significant figures. Compare with the vacuum values.

Mathematica hints: I started this by typing in Equation (2.39) as it appears in the book and giving this equation the name "eq1". Since Mathematica's built-in functions and variables begin with capital letters, all of my own variables start with lowercase letters.

```
eq1 = (vy0 + vter)*r/vx0 + vter*tau*Log[1-r/(vx0*tau)]==0
```

Then I defined "eq2" to be the same equation with a few handy replacements, using Mathematica's `ReplaceAll` operator, whose shorthand is `/.` (slash dot), which when I read it sounds like "such that."

```
eq2 = eq1 /. {tau->vter/g, vx0->v0*Cos[th], vy0->v0*Sin[th]}
```

Then I defined "eq3" to be eq2 with a few more replacements:

```
eq3 = eq2 /. {v0->1, vter->1, g->1}
```

which Mathematica then writes as

```
Log[1 - r*Sec[th]] + r*Sec[th]*(1 + Sin[th]) == 0
```

To solve this for $\theta = 0.75$, I do one more replacement and use the `Solve` function: `Solve[eq3 /. th->0.75]` which finds $r = 0.499597$, which you can check by plugging in numbers. (I take only the left-hand side of "eq3" by taking the "`First`" element of the equation.)

```
First[eq3] /. {th->0.75, r->0.4996}
```

You can repeat the `Solve` step for other values of θ. You might also want to check that using `vter->1000` gives you approximately the range you calculated (in these funny units) for part (a). By the way, if you are already a Mathematica expert and you know a more straightforward (but still understandable by a beginner) way of solving this problem, please send it to me!

XC3*. Optional/extra-credit. A ball is thrown with initial speed v_0 up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. (a) Write down Newton's second law using these axes and find the ball's position as a function of time. (b) Show that the ball lands a distance R from its launch point, where $R = 2v_0^2 \sin\theta \cos(\theta+\phi)/(g\cos^2\phi)$. (c) Show that for a given v_0 and ϕ, the maximum possible range up the inclined plane is $R_{\max} = v_0^2/[g(1+\sin\phi)]$. (d) For level ground, it is well known that the maximum range occurs for a projectile thrown at 45°. Can you give a simple statement of what angle corresponds to the maximum range for the projectile on an incline?

XC4*. Optional/extra-credit. A cannon shoots a ball at an angle θ above the horizontal ground. (a) Neglecting air resistance, use Newton's second law to find the ball's position as a function of time. (Use axes with x measured horizontally and y measured vertically.) (b) Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight? [Hint: Using your solution to part (a), you can write down $r^2 = x^2 + y^2$, and then find the condition that r^2 is always increasing.]

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