Physics 351, Spring 2015, Homework #2.

Due at start of class, Friday, January 30, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
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to tell me how the homework went for you.

1. Two elephants, each of mass m, are standing at one end of a stationary railroad flatcar of mass M, which has frictionless wheels. Either elephant can run to the other end of the flatcar and jump off with the same speed u (relative to the car). (a) Use conservation of momentum to find the speed of the recoiling car if the two elephants run and jump simultaneously. (b) What is it if the second elephant starts running only after the first has already jumped? Which procedure gives the greater speed to the car? Hint: The speed u is the speed of either elephant, *relative to the car*, just after it has jumped; it has the same value for either elephant and is the same in parts (a) and (b).

2. Many applications of conservation of momentum involve conservation of energy as well. Consider an elastic collision between two equal-mass bodies, one of which is initially at rest. Let their velocities be $\mathbf{v}_1$ and $\mathbf{v}_2 = 0$ before the collision, and $\mathbf{v}_1'$ and $\mathbf{v}_2'$ after. Write down the vector equation representing conservation of momentum and the scalar equation which expresses that the collision is elastic. Use these to prove that the angle between $\mathbf{v}_1'$ and $\mathbf{v}_2'$ is 90°.

3. The first couple of minutes of the launch of a space shuttle can be described very roughly as follows: The initial mass is 2×10^6 kg, the final mass (after 2 minutes) is about 1×10^6 kg, the average exhaust speed v_{ex} is about 3000 m/s, and the initial velocity is, of course, zero. If all this were taking place in outer space, with negligible gravity, what would be the shuttle's speed at the end of this stage? What is the thrust during the same period and how does it compare with the initial total weight of the shuttle (on earth)?

4. (a) Consider a rocket traveling in a straight line subject to an external force F^{ext} acting along the same line. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}.$$

(b) Specialize to the case of a rocket taking off vertically (from rest) in a gravitational field g, so the equation of motion becomes

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg.$$

Assume that the rocket ejects mass at a constant rate, $\dot{m} = -k$ ($k > 0$), so that $m = m_0 - kt$. Solve Eq. (3.30) for v as a function of t, using separation of variables. (c) Using the rough data from Problem 3, find the space shuttle's speed two minutes into flight, assuming (what is nearly true) that it travels vertically up during this period and that g doesn't change appreciably. Compare with the corresponding result if there were no gravity. (d) Describe what would happen to a rocket that was designed so that the first term on the right of Eq. (3.30) was smaller than the initial value of the second.

5. Start from the result $v(t)$ from Problem 4b. Now integrate $v(t)$ and show that the rocket's height as a function of t is

$$y(t) = v_{\rm ex} t - \frac{1}{2} g t^2 - \frac{m v_{\rm ex}}{k} \ln(\frac{m_0}{m}).$$

Using the numbers given in Problem 3, estimate the space shuttle's height after two minutes.

6. (a) We know that the path of a projectile thrown from the ground is a parabola (if we ignore air resistance). In light of Eq. (3.12), what would be the subsequent path of the CM of the pieces if the projectile exploded in midair? (b) A shell is fired from level ground so as to hit a target 100 m away. Unluckily the shell explodes prematurely and breaks into two equal pieces. The two pieces land at the same time, and one lands 100 m beyond the target. Where does the other piece land? (c) Is the same result true if they land at different times (with one piece still landing 100 m beyond the target)?

7. Use spherical polar coordinates r, θ, ϕ to find the CM of a uniform solid hemisphere of radius R, whose flat face lies in the xy plane with its center at the origin. Before you do this, you will need to convince yourself that the element of volume in spherical polars is $\mathrm{d}V = r^2 \ \mathrm{d}r \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi$.

8. A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. (a) Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle. What is the particle's angular velocity now? (b) Now let's see what happens during the pull described in part (a). Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0. Starting at $t = 0$, I pull the string with constant velocity v so that the radial distance (r) to the mass decreases. Draw a force diagram for the mass and find a differential equation for $\omega(t)$. Find $\omega(t)$ and also find the force $F(t)$ that I need to exert on the string. [Hint: one component of the force exerted on m by the string is always zero.]

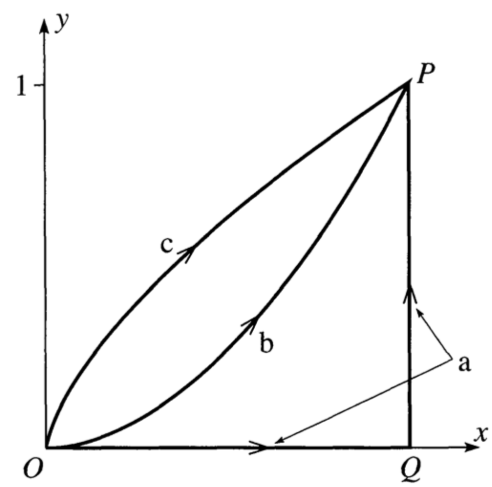
9. Show that the moment of inertia of a uniform solid sphere rotating about a diameter is $\frac{2}{5}MR^2$. The integral is easiest in spherical polar coordinates, with the axis of rotation taken to be the z axis.

10. Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at angle γ to the horizontal. The instantaneous point of contact between the disk and the incline is called P. (a) Draw a free-body diagram, showing all forces on the disk. (b) Find the linear acceleration $\dot{v}$ of the disk by applying the result $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ for rotation about P. (Remember to use the parallel-axis theorem for rotation about a point on the circumference.) (c) Derive the same result by applying $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ to the rotation about the CM. (In this case there will be an extra unknown, the force of friction, which you can eliminate using the equation of motion of the CM.)

11. Evaluate the work done

$$W = \int_O^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_O^P (F_x \, \mathrm{d}x + F_y \, \mathrm{d}y)$$

by the two-dimensional force $\mathbf{F} = (x^2, 2xy)$ along the three paths joining the origin to the point $P = (1, 1)$ as shown in the figure and defined as follows: (a) This path goes along the x axis to $Q = (1, 0)$ and then straight up to P. (b) On this path $y = x^2$, and you can write $\mathrm{d}y = 2x \, \mathrm{d}x$. (c) This path is given parameterically as $x = t^3$, $y = t^2$. In this case, convert the integral into an integral over t.



12. Near to the point where I am standing on the surface of Planet X, the gravitational force on a mass m is vertically down but has magnitude $m\gamma y^2$ where γ is a constant and y is the mass's height above the horizontal ground. (a) Find the work done by gravity on a mass m moving from $\mathbf{r}_1$ to $\mathbf{r}_2$, and use your answer to show that gravity on Planet X, although most unusual, is still conservative. Find the corresponding potential energy. (b) Still on the same planet, I thread a bead on a curved, frictionless, rigid wire, which extends from ground level to a height h above the ground. Show clearly in a picture the forces on the bead when it is somewhere on the wire. (Just name the forces so it's clear what they are; don't worry about their magnitude.) Which of the forces are conservative and which are not? (c) If I release the bead from rest at a height h, how fast will it be going when it reaches the ground?

13. Consider a small frictionless puck perched at the top of a fixed sphere of radius R. If the puck is given a tiny nudge so that it begins to slide down, through what vertical height will it descend before it leaves the surface of the sphere? [Hint: At what value of the normal force between sphere and puck does the puck leave the sphere?]

14. Use the property (4.35) of the gradient to prove the following: (a) The vector ∇f at any point $\mathbf{r}$ is perpendicular to the surface of constant f through $\mathbf{r}$. (What is $\mathrm{d}f$ for a small displacement $\mathrm{d}\mathbf{r}$ that lies in a surface of constant f?) (b) The direction of ∇f at any point $\mathbf{r}$ is the direction in which f increases fastest as we move away from $\mathbf{r}$. (Choose a small displacement $\mathrm{d}\mathbf{r} = \epsilon\mathbf{u}$, where $\mathbf{u}$ is a unit vector and ϵ is fixed and small. Find the direction of $\mathbf{u}$ for which the corresponding $\mathrm{d}f$ is maximum, bearing in mind that $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$.)

15. Which of the following forces is conservative? (a) $\mathbf{F} = k\ (x, 2y, 3z)$ where k is a constant. (b) $\mathbf{F} = k\ (y, x, 0)$. (c) $\mathbf{F} = k\ (-y, x, 0)$. For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\mathbf{F} = -\nabla U$.

16. Consider a mass m on the end of a spring of Hooke's-law constant k and constrained to move along the horizontal x axis. If we place the origin at the spring's equilibrium position, the potential energy is $\frac{1}{2}kx^2$. At time $t = 0$ the mass is sitting at the origin and is given a sudden kick to the right so that it moves out to a maximum displacement at $x_{\mathrm{max}} = A$ and then continues to oscillate about the origin. (a) Write down the equation for conservation of energy and solve it to give the mass's velocity $\dot{x}$ in terms of the position x and the total energy E. (b) Show that $E = \frac{1}{2}kA^2$, and use this to eliminate E from your expression for $\dot{x}$. Use the result (4.58), $t = \int \mathrm{d}x'/\dot{x}(x')$, to find the time for the mass to move from the origin out to a position x. (c) Solve the result of part (b) to give x as a function of t and show that the mass executes simple harmonic motion with period $2\pi\sqrt{m/k}$.

17. A block rests on a wedge whose incline has coefficient of static friction μ and is at angle θ from the horizontal. (See figure at top of next page.) (a) Assuming that the wedge is fixed in position, find the maximum value of θ such that the block remains motionless on the wedge. (b) Now suppose that $\tan\theta > \mu$, so that the block slides downhill if the wedge is motionless. Also suppose that **the wedge** is accelerating to the right with constant acceleration a. Find the minimum and maximum values of a for which the block can remain motionless w.r.t. the wedge.

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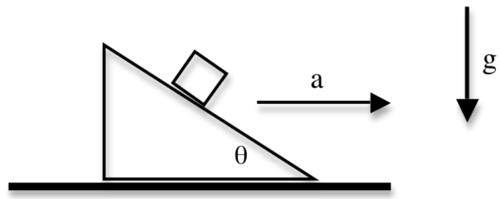
(extra credit on following pages)

Figure for problem 17: 

XC1*. Optional/extra-credit. A grenade is thrown with initial velocity $\mathbf{v}_0$ from the origin at the top of a high cliff, subject to negligible air resistance. (a) Using Mathematica (or your favorite alternative), plot the orbit, with the following parameters: $\mathbf{v}_0 = (4, 4)$, $g = 1$, and $0 \le t \le 4$ (and with x measured horizontally and y vertically up). Add to your plot suitable marks (dots or crosses, for example) to show the positions of the grenade at $t = 1, 2, 3, 4$. (b) At $t = 4$, when the grenade's velocity is $\mathbf{v}$, it explodes into two equal pieces, one of which moves off with velocity $\mathbf{v} + \Delta\mathbf{v}$. What is the velocity of the other piece? (c) Assuming that $\Delta\mathbf{v} = (1, 3)$, add to your original plot the paths of the two pieces for $4 \le t \le 9$. Insert marks to show their positions at $t = 5, 6, 7, 8, 9$. Find some way to show clearly that the CM of the two pieces continues to follow the original parabolic path.

XC2*. Optional/extra-credit. A system consists of N masses m_α at positions $\mathbf{r}_\alpha$ relative to a fixed origin O. Let $\mathbf{r}'_\alpha$ denote the position of m_α relative to the CM; that is, $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$. (a) Make a sketch to illustrate this last equation. (b) Prove the useful relation that $\sum m_\alpha \mathbf{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious? (c) Use this relation to prove the result (3.28) that the rate of change of the angular momentum *about the CM* is equal to the total external torque about the CM. (This result is surprising since the CM may be accelerating, so that it is not necessarily a fixed point in any inertial frame.)

XC3*. Optional/extra-credit. [Computer] A mass m confined to the x axis has potential energy $U = kx^4$ with $k > 0$. (a) Sketch this potential energy and qualitatively describe the motion if the mass is initially stationary at $x = 0$ and is given a sharp kick to the right at $t = 0$. (b) Use (4.58) to find the time for the mass to reach its maximum displacement $x_{\max} = A$. Give your answer as an integral over x in terms of m, A, and k. Hence find the period τ of oscillations of amplitude A as an integral. (c) By making a suitable change of variables in the integral, show that the period τ is inversely proportional to the amplitude A. (d) The integral of part (b) cannot be evaluated in terms of elementary functions, but it can be done numerically. Find the period for the case that $m = k = A = 1$.

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