Physics 351, Spring 2015, Homework #3.

Due at start of class, Friday, February 6, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
`positron.hep.upenn.edu/q351`
to tell me how the homework went for you.

1. Problems 1 and 2 are (embarrassingly easy) problems from Chapter 2 that I think are worth doing because they walk you through things that are worth knowing about air resistance. (I forgot to include them on HW #1.) The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. (a) Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ρ, show that the rate at which the projectile encounters fluid (mass/time) is $\rho A v$. (b) Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is $\rho A v^2$. (c) More realistically, as it turns out, the force takes the form $\boxed{f_{\rm quad} = \kappa \rho A v^2}$ where $\kappa < 1$ depends on the shape of the projectile. Show that the boxed equation reproduces $f_{\rm quad} = cv^2 = \gamma D^2 v^2$, where the density of air at STP is $\rho = 1.29\ \mathrm{kg/m^3}$ and given that $\kappa = 1/4$ for a sphere. Check that you reproduce the textbook's value $\gamma = 0.25\ \mathrm{N\cdot s^2/m^4}$.

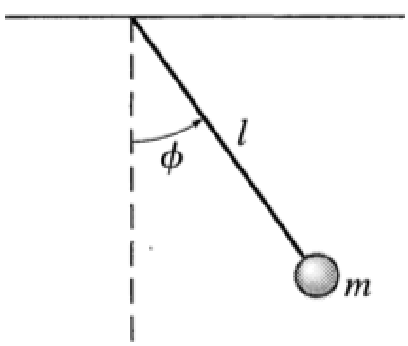
2. (a) The origin of the linear drag force on a sphere in a fluid is the viscosity of the fluid. According to Stokes's law, the viscous drag on a sphere is $f_{\rm lin} = 3\pi\eta D v$ where η is the viscosity[1] of the fluid, D is the sphere's diameter, and v its speed. Given the viscosity of air at STP, $\eta = 1.7\times 10^{-5}\ \mathrm{N\cdot s/m^2}$, show that this expression reproduces $f_{\rm lin} = bv = \beta D v$, where $\beta = 1.6\times 10^{-4}\ \mathrm{N\cdot s/m^2}$. (b) The quadratic drag force on a moving sphere in a fluid is given by the boxed equation in Problem 1. Show that the ratio of drag forces can be written as $f_{\rm quad}/f_{\rm lin} = R/48$, where the dimensionless *Reynolds number*[2] is $R = Dv\rho/\eta$, where ρ is the fluid's density. Clearly the Reynolds number is a measure of the relative importance of the two kinds of drag.

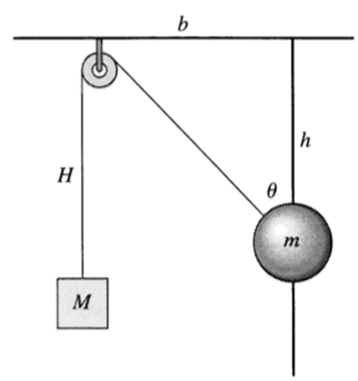
[1] To define viscosity η, imagine a wide channel along which fluid is flowing (x direction) such that the velocity v is zero at the bottom ($y = 0$) and increases toward the top ($y = h$), so that successive layers of fluid slide across one another with a velocity gradient $\mathrm{d}v/\mathrm{d}y$. The force F with which an area A of any one layer drags the fluid above it is proportional to A and to $\mathrm{d}v/\mathrm{d}y$, and η is defined as the constant of proportionality: $F = \eta A \mathrm{d}v/\mathrm{d}y$.

[2] The factor 1/48 is for a sphere.

3. (a) Verify the expression (Eq. 4.59) for the potential energy of the cube balanced on a cylinder in Example 4.7 (page 130). [Hint: To understand the $r\theta$ factor, imagine the cylinder rolling on the cube.] (b) Make graphs of $U(\theta)$ for $b = 0.9r$ and $b = 1.1r$, preferably by computer. (For simplicity, choose units such that r, m, and g all equal 1.) (c) Use your graphs to confirm the findings of Example 4.7 concerning the stability of the equilibrium at $\theta = 0$. Are there any other equilibrium points, and are they stable?

4. An interesting one-dimensional system is the simple pendulum, consisting of a point mass m fixed to the end of a massless rod (length l), as shown in the left figure below. The pendulum's position can be specified by its angle ϕ from the equilibrium position. (a) Prove that the pendulum's potential energy is $U(\phi) = mgl(1 - \cos\phi)$. Then write down the total energy E as a function of ϕ and $\dot{\phi}$. (b) Show that requiring the total energy E to be independent of time ($\mathrm{d}E/\mathrm{d}t = 0$) gives the equation of motion for ϕ, and that this EOM is just the familiar $\Gamma = I\alpha$, where Γ is torque, I is moment of inertia, and $\alpha = \ddot{\phi}$. (c) Assuming that $\phi(t) \ll 1$, solve for $\phi(t)$ and show that the motion is periodic with period $\tau_0 = 2\pi\sqrt{l/g}$.

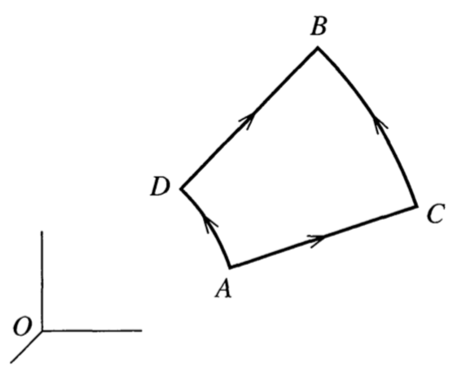


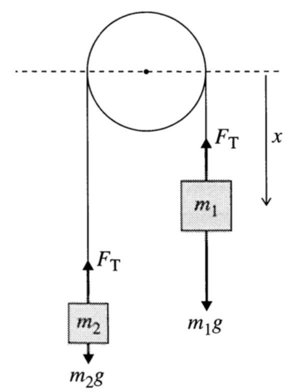


5. A metal ball (mass m) with a hole through it is threaded on a frictionless vertical rod. A massless string (total length l) attached to the ball runs over a massless, frictionless pulley and supports a block of mass M, as shown in the right figure above. The positions of the two masses can be specified by the one angle θ. (a) Write down the potential energy $U(\theta)$. (To get $U(\theta)$, eliminate heights H and h in favor of θ, b, and l, assuming the pulley and ball have negligible size.) (b) By differentiating $U(\theta)$, find whether the system has an equilibrium position, and for what values of m and M equilibrium can occur. Discuss the stability of any equiibrium positions.

6. Section 4.8 claims that a force $\vec{F}(\vec{r})$ that is central and spherically symmetric is automatically conservative. Here are two ways to prove it. (a) Since $\vec{F}(\vec{r})$ is central and spherically symmetric, it must have the form $\vec{F}(\vec{r}) = f(r)\hat{r}$. Using Cartesian coordinates, show that this implies that $\nabla \times \vec{F} = 0$. (b) Even quicker, using the expression given inside the textbook's back cover for $\nabla \times \vec{F}$ in spherical polar coordinates, show that $\nabla \times \vec{F} = 0$.

7. Problem 6 suggests two proofs that a central, spherically symmetric force is automatically conservative, but neither proof makes really clear *why* this is so. Here is a proof that is less complete but more insightful. Consider any two points A and B and two different paths ACB and ADB connecting them, as shown in the left figure below. Path ACB goes radially out from A until it reaches the radius r_B of B, and then around a sphere (center O) to B. Path ADB goes around a sphere of radius r_A until it reaches the line OB, and then radially out to B. Explain clearly why the work done by a central, spherically symmetric force $\vec{F}$ is the same along both paths. (This doesn't prove that the work is the same along *any* two paths from A to B. If you want you can complete the proof by showing that any path can be approximated by a series of paths moving radially in or out, combined with paths of constant r.)



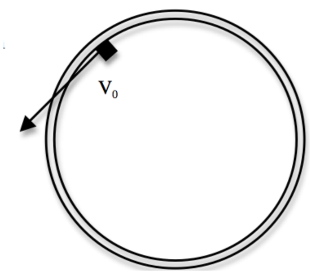


8. Consider the Atwood machine shown in the right figure above, where the pulley has radius R and moment of inertia I. (a) Write down the total energy of the two masses and the pulley in terms of the coordinate x and $\dot{x}$. (Remember the K.E. of the spinning wheel.) (b) Show (as is true for any conservative one-dimensional system) that you can obtain the EOM for x by differentiating the equation $E = \text{const}$. Check that the EOM is the same as you would obtain by applying Newton's second law separately to the two masses and the pulley, and then eliminating the two unknown tensions from the three resulting equations. (This problem seems to be hinting toward the notion that writing down expressions for energies can lead us straightforwardly to the equations of motion — as we'll see in the Lagrangian formulation.)

9. A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U = kr^n$. (a) Prove the **virial theorem**, that $T = nU/2$. (b) What does the virial theorem (assuming that it generalizes beyond circular orbits) imply for $n = -1$ (the gravitational Kepler problem) and for $n = 2$ (the harmonic-oscillator problem)?

10. A chain of mass M and length L is suspended vertically with its lowest end just barely touching a scale. The chain is released and falls onto the scale. What is the reading on the scale when a length x of the chain has fallen? [Hint: The reading on the scale equals the normal force exerted by the scale on the chain. Consider the motion of the center-of-mass of the chain. The maximum reading is $3Mg$.]

11. A block of mass m slides on a frictionless (horizontal) table and is constrained to move along the inside of a ring of radius R, which is fixed to the table. At $t = 0$ the mass is moving (tangentially) along the inside of the ring with velocity v_0. The coefficient of kinetic friction between the block and ring is μ. Find the velocity $\dot{s}$ and position s (the arc length traveled) of the block as a function of time.



12. The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

for $0 < r < \infty$, with U_0, R, and λ all positive constants. Find the equilibrium position r_0. Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const.} + \frac{1}{2}kx^2$. What is the natural angular frequency ω_0 for small oscillations?

13. Another interpretation of the Q of a resonance comes from the following: Consider the motion of a driven damped oscillator after any transients have died out, and suppose that it is being driven close to resonance, so that you can set $\omega = \omega_0$. (a) Show that the oscillator's total energy $(T+U)$ is $E = \frac{1}{2}m\omega^2 A^2$. (b) Show that the energy ΔE_{dis} dissipated during one cycle by the damping force F_{damp} is $2\pi m \beta \omega A^2$. (Remember power is Fv.) (c) Hence show that Q is 2π times the ratio $E/\Delta E_{\text{dis}}$.

14. You can make the Fourier series solution for a periodically driven oscillator a bit tidier if you don't mind using complex numbers. Obviously the periodic force of (Eq. 5.90) can be written as $f = \text{Re}(g)$, where the complex function

g is $g(t) = \sum_{n=0}^{\infty} f_n e^{in\omega t}$. Show that the real solution for the oscillator's motion can likewise be written as $x = \mathrm{Re}(z)$, where $z(t) = \sum_{n=0}^{\infty} C_n e^{in\omega t}$ and $C_n = f_n/(\omega_0^2 - n^2\omega^2 + 2i\beta n\omega)$. This solution avoids our having to worry about the real amplitude A_n and phase shift δ_n separately.

15. Show that the path $y = y(x)$ for which the integral $\int_{x_1}^{x_2} x\sqrt{1-(y')^2}\,\mathrm{d}x$ is stationary is an arcsinh function.

16. Find the path $y = y(x)$ for which the integral $\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+(y')^2}\;\mathrm{d}x$ is stationary.

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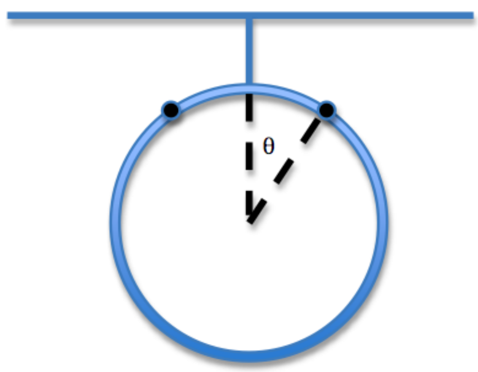
XC1*. Optional/extra-credit. [Computer] Consider the simple pendulum of Problem 4. You can get an expression for the pendulum's period (good for both large and small oscillations) using the method of (Eq. 4.57), as follows: (a) Using $U(\phi) = mgl(1-\cos\phi)$, find $\dot{\phi}$ as a function of ϕ. Next use $t = \int \mathrm{d}\phi/\dot{\phi}$ to write the time for the pendulum to travel from $\phi = 0$ to its maximum value Φ, and use this to show that the period of oscillation is

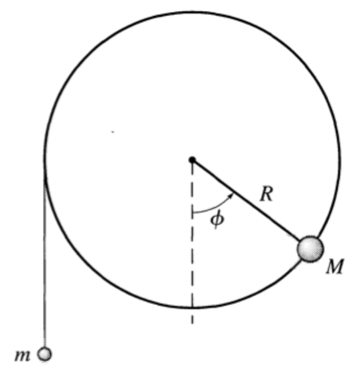
$$\tau = \frac{\tau_0}{\pi}\int_0^{\Phi} \frac{\mathrm{d}\phi}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}} = \frac{2\tau_0}{\pi}\int_0^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}\sqrt{1-A^2u^2}}$$

where $\tau_0 = 2\pi\sqrt{l/g}$. (Use substitution $\sin(\phi/2) = Au$, where $A = \sin(\Phi/2)$.) These integrals cannot be evaluated in terms of elementary functions, but the second integral is a standard integral called the *complete elliptic integral of the first kind,* sometimes denoted $K(A^2)$, whose values can be looked up or calculated with Mathematica's `EllipticK(A^2)`. (b) Use Mathematica (or your favorite software) to make a graph of τ/τ_0 vs. amplitude Φ, for $0 \le \Phi \le 3$ radians and comment. Explain what happens to τ (and why!) as $\Phi \to \pi$.

XC2*. Optional/extra-credit. If you have not already done so, do XC1(a). (a) If the amplitude Φ is small, then so is $A = \sin(\Phi/2)$. If the amplitude is very small, we can simply ignore the last square root in the integral in (XC1). Show that this gives the familiar result $\tau = \tau_0 = 2\pi\sqrt{l/g}$. (b) If the amplitude is small but not very small, we can improve on the approximation of part (a). Use the binomial expansion to give the approximation $1/\sqrt{1-A^2u^2} \approx 1 + \frac{1}{2}A^2u^2$ and show that, in this limit, $\tau \approx \tau_0[1 + \frac{1}{4}\sin^2(\Phi/2)]$. (c) What percentage correction does the second term represent for an amplitude of 45°? (The exact answer for $\Phi = 45°$ is 1.040 τ_0 to four significant figures.)

XC3*. Optional/extra-credit. A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction, as shown in the left figure below. The beads are released simultaneously from rest (given an infinitesimal kick) at the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > \frac{3}{2}M$, and find the angle θ at which this occurs. [Hint: If $M = 0$, then $\cos\theta = \frac{2}{3}$.] You will receive partial extra-credit if you do the problem assuming $M = 0$, but for full credit, you must account for the mass M of the ring.





XC4*. Optional/extra-credit. The right figure above shows a massless wheel of radius R, mounted on a frictionless horizontal axle. A point mass M is glued to the edge of the wheel, and a mass m hangs from a string wrapped around the perimeter of the wheel. (a) Write down the total PE of the two masses as a function of the angle ϕ. (b) Use this to find the values of m/M for which there are any positions of equilibrium. Describe the equilibrium positions, discuss their stability, and explain your answers in terms of torques. (c) Graph $U(\phi)$ for the cases $m = 0.7M$ and $m = 0.8M$, and use your graphs to describe the behavior of the system if I release it from rest at $\phi = 0$. (If the system oscillates, you **do not** need to find the frequency of oscillation.) (d) Find the critical value of m/M such that if $\frac{m}{M} < (\frac{m}{M})_{\text{crit}}$, the system oscillates, while if $\frac{m}{M} > (\frac{m}{M})_{\text{crit}}$ it does not (if released from rest at $\phi = 0$).

XC5*. Optional/extra-credit. Repeat the calculations of Example 5.3 (page 185) with all the same parameters, but with the initial conditions $x_0 = 2$ and $v_0 = 0$. Graph $x(t)$ for $0 \le t \le 4$ and compare with the graph of Example 5.3. Explain the similarities and differences, e.g. for what region in time do the two graphs differ appreciably?

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