Physics 351, Spring 2015, Homework #4.

Due at start of class, Friday, February 13, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
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to tell me how the homework went for you.

1. A mass m is fixed to a given point on the rim of a wheel of radius R that rolls without slipping on the ground. The wheel is massless, except for a mass M located at its center. Using Lagrange's equation, find the EOM for the angle through which the wheel rolls. For the case where the wheel undergoes small oscillations, find the frequency.

2. The shortest path between two points on a *curved surface*, such as the surface of a sphere, is called a **geodesic**. To find a geodesic, one has first to set up an integral that gives the length of a path on the surface in question. This will always be similar to (Eq. 6.2) but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical polar coordintes (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + (\phi')^2 \, \sin^2(\theta)} \; \mathrm{d}\theta$$

where $\phi' = \mathrm{d}\phi/\mathrm{d}\theta$ and we assume that the path is expressed as $\phi = \phi(\theta)$ with end points specified as (θ_1, ϕ_1) and (θ_2, ϕ_2).

3. Use the result of Problem 2 to prove that the geodesic (shortest path) between two given points on a sphere is a great circle. [Hint: The integrand $f(\phi, \phi', \theta)$ is independent of ϕ, so the Euler-Lagrange equation reduces to $\partial f/\partial\phi' = c$, a constant. This gives you ϕ' as a function of θ. You can avoid doing the final integral by the following trick: There is no loss of generality in choosing your z axis to pass through the point 1. Show that with this choice the constant c is necessarily zero, and describe the corresponding geodesics.]

4. In general the integrand $f(y, y', x)$ whose integral we wish to minimize depends on y, y', and x. There is a considerable simplification if f happens to be independent of y, that is, $f = f(y', x)$. Prove that when this happens, the E-L equation reduces to $\partial f/\partial y' = \text{const}$. Since this is a first-order differential equation for $y(x)$, while the E-L equation is generally second-order, this is an important simplification that is sometimes called a **first integral** of the E-L equation. In Lagrangian mechanics, this simplification arises when a component of momentum is conserved.

5. Here is a second situation in which you can find a "first integral" of the E-L equation: Argue that if it happens that the integrand $f(y, y', x)$ does not depend explicitly on x, that is, $f = f(y, y')$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial y}\, y' + \frac{\partial f}{\partial y'}\, y''$$

Use the E-L equation to replace $\partial f/\partial y$ on the right, and hence show that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\,\frac{\partial f}{\partial y'}\right)$$

This gives you a first integral

$$f - y'\,\frac{\partial f}{\partial y'} = \text{const}$$

This can simplify several calculations. In Lagrangian mechanics, where the independent variable is the time t, the corresponding result is that if the Lagrangian function is independent of t, then energy is conserved, and the expression that equals the constant energy is called the *Hamiltonian*.

6. Assume that the speed of light in a given slab of material is proportional to the height above the base of the slab. (Since the speed of light in a material equals c/n, you could equivalently say that as a function of height y, the index of refraction n is given by $n(y) = y_0/y$.) Show that light moves in circular arcs in this material, by assuming that light takes the path of least time between two points (Fermat's "principle of least time").

7. Use the E-L equation to find the shortest path between two points in a plane using polar coordinates. Show that the answer is $r\cos(\phi + \alpha) = C$, where α and C are constants. Make a graph using Mathematica's `PolarPlot` (or equivalent) to convince yourself that this is in fact the equation for a straight line.

8. The gravitational PE of a cable (or a chain), of mass per unit arc-length μ, hanging symmetrically between two utility poles located at $x = \pm L$ is

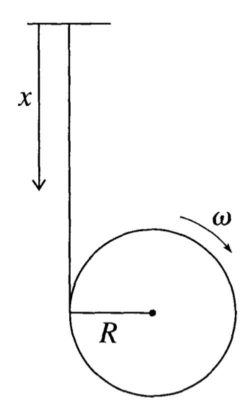
$$U[y] = \int \mu g y \, \mathrm{d}s = \mu g \int_{x=-L}^{L} y \, \sqrt{1 + (y')^2} \, \mathrm{d}x$$

where $y(x)$ is the cable's height vs. horizontal position. In equilibrium, the cable's shape $y(x)$ is that which minimizes the potential energy U. Use the integrated form of the E-L equation from Problem 5 to show that the functional form of $y(x)$ is a catenary, $y = B\cosh((x - A)/B)$, where A and B are constants. It may help to notice that

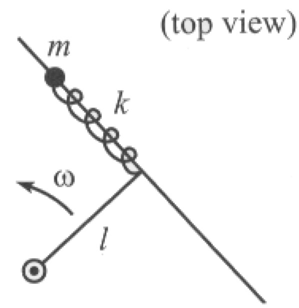
$$\frac{\mathrm{d}}{\mathrm{d}y} B \cosh^{-1}\left(\frac{y}{B}\right) = \frac{1}{\sqrt{(y/B)^2 - 1}}$$

9. (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal mass $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. [Here x is the extension of the spring, $x = (x_1 - x_2 - l)$, where l is the spring's unstretched length, and we assume that $x_1 > x_2$.] (b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1 + x_2)$ (the CM position) and x (the extension), and write down the two Lagrange equations for X and x. (c) Solve for $X(t)$ and $x(t)$ and describe the motion.

10. In a crude model of a yoyo, shown in the figure, a massless string is suspended vertically from a fixed point and the other end is wrapped several times around a uniform cylinder of mass m and radius R. When the cylinder is released it moves vertically down, rotating as the string unwinds. Write down the Lagrangian, using the distance x as your generalized coordinate. Find the Lagrange EOM and show that the cylinder accelerates downward with $\ddot{x} = 2g/3$. [Hint: after writing down $T - U$, where T includes both translational and rotational K.E., rewrite ω in terms of $\dot{x}$.]

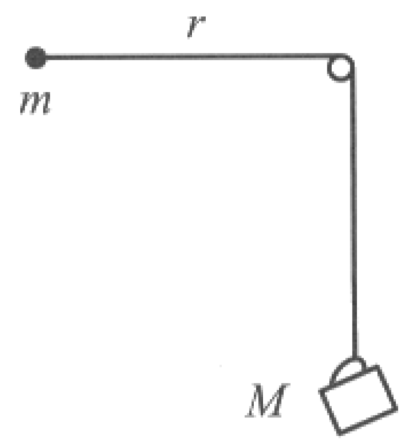


11. A rigid T consists of a long rod glued perpendicular to another rod of length l that is pivoted at the origin. The T rotates around in a horizontal plane with constant frequency ω. A mass m is free to slide along the long rod and is connected to the intersection of the rods by a spring with spring constant k and relaxed length zero (see figure). Find $r(t)$, where r is the position of the mass along the long rod. There is a special value of ω. What is it, and why is it special?



12. Consider the setup in Problem 11, but now let the T swing around in a vertical plane with constant frequency ω. Find $r(t)$. There is a special value of ω. What is it, and why is it special? (You may assume $\omega < \sqrt{k/m}$.)

13. A coffee cup of mass M is connected to a mass m by a string. The coffee cup hangs over a frictionless pulley of negligible size, and the mass m is initially held with the string horizontal, as shown in the figure. The mass m is then released. (a) Find the EOM for r (the length of string between m and the pulley) and θ (the angle that the string to m makes with the horizontal). Assume that m somehow doesn't run into the string holding the cup up. The coffee cup will initially fall, but it turns out that it will reach a lowest point and then rise back up. (b) Use Mathematica (or similar) to determine numerically the ratio of the r at this lowest point to the r at the start, as a function of the value of m/M. (To check your computation, a value of $m/M = 1/10$ yields a ratio of about 0.208.)



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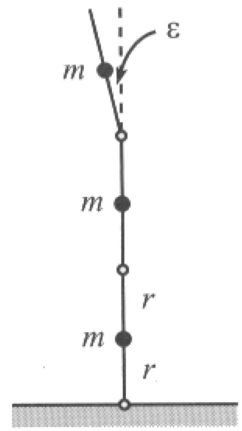
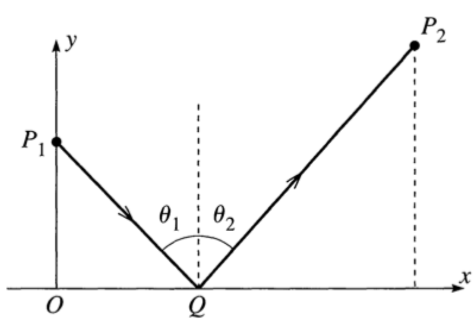
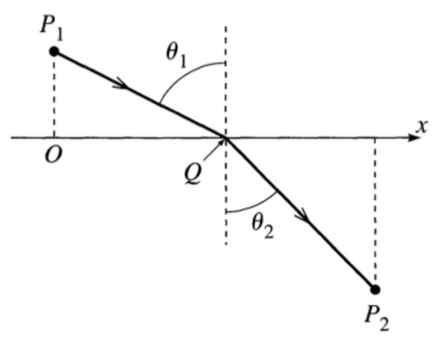
XC1*. Optional/extra-credit. Consider a medium in which the refractive index n is inversely proportional to r^2, i.e. $n = a/r^2$, where r is the distance from the origin. Use Fermat's principle of least time to find the path of a ray of light traveling in a plane containing the origin. [Hint: use 2D polar coordinates and write the path as $\phi = \phi(r)$. The Fermat integral will have the ϕ-independent form of Problem 4, so you can solve for ϕ' and integrate to get $\phi(r)$. Rewrite this to give $r(\phi)$ and show that the resulting path is a circle through the origin.] Discuss the progress of the light around the circle. Does the light ever actually reach the origin?

XC2*. Optional/extra-credit. Three massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at their ends, as shown in the figure. The bottom end of the lower stick is hinged at the ground. They are held such that the lower two sticks are vertical, and the upper one is tilted at a small angle ε w.r.t. the vertical. They are then released. At this instant, what are the angular accelerations of the three sticks? Work in the approximation where ε is very small. (If you need a hint, the two-stick version is worked out as Morin's Problem 6.2.)



XC3*. Optional/extra-credit. Consider a ray of light traveling in a vacuum from point P_1 to P_2 by way of the point Q on a plane mirror, as in the left figure below. Show that Fermat's principle of least time implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, that $\theta_1 = \theta_2$. [Hints: Let the mirror lie in the xz plane, and let P_1 lie on the y axis at $(0, y_1, 0)$ and P_2 in the xy plane at $(x_2, y_2, 0)$. Finally let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]

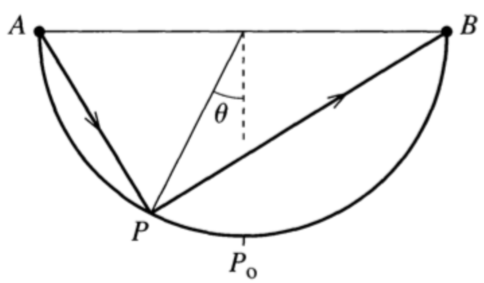


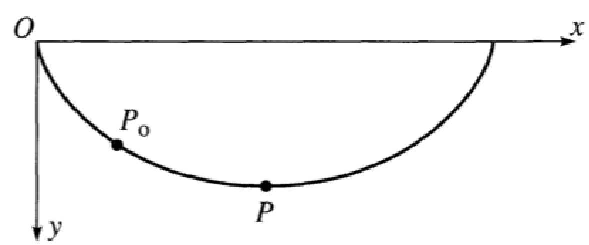


XC4*. Optional/extra-credit. A ray of light travels from point P_1 in a medium of refractive index n_1 to P_2 in a medium of index n_2, by way of the point Q on the plane interface between the two media, as in the right figure

above. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys Snell's law, that $n_1 \sin\theta_1 = n_2 \sin\theta_2$. [Hints: Let the interface be the xz plane, and let P_1 lie on the y axis at $(0, h_1, 0)$ and P_2 in the xy plane at $(x_2, -h_2, 0)$. Finally let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies Snell's law.]

XC5*. Optional/extra-credit. Fermat's principle is often stated as, "the travel time of a ray of light, moving from point A to B, is minimum along the actual path." Strictly speaking it should say that the time is *stationary*, not minimum. In fact one can construct situations for which the time is **maximum** along the actual path. Here is one: Consider the concave, hemispherical mirror shown in the left figure below, with A and B at opposite ends of a diameter. Consider a ray of light traveling in a vacuum from A to B with one reflection at P, in the same vertical plane as A and B. According to the law of reflection, the actual path goes via point P_0 at the bottom of the hemisphere ($\theta = 0$). Find the time of travel along the path APB as a function of θ and show that it is *maximum* at $P = P_0$. This shows the time is maximum w.r.t. paths of the form APB with just two straight segments. It is easy to see that it is *minimum* for other kinds of path, so the correct general statement is that it is *stationary* for arbitrary variations of the path.





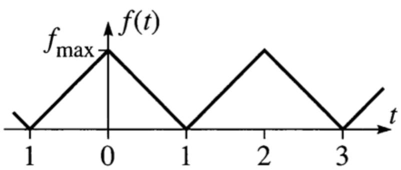
XC6*. Optional/extra-credit. Consider a single loop of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ (Eq. 6.26) with a fixed value of a, as shown in the right figure above. A cart is released from rest at point P_0 anywhere on the track between O and the lowest point P (that is, P_0 has parameter $0 < \theta_0 < \pi$). Show that the time for the cart to roll from P_0 to P is given by the integral

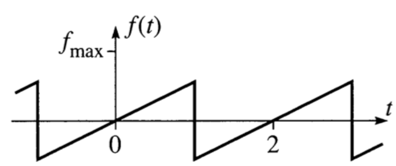
$$\text{time}(P_0 \to P) = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}}\, \mathrm{d}\theta$$

and prove that this time is equal to $\pi\sqrt{a/g}$. Since this is *independent of the* *position of* P_0, the cart takes the same time to roll from P_0 to P, whether P_0 is at O, or anywhere between O and P, even infinitesimally close to P. Explain qualitatively how this surprising result can possibly be true. [Hint: To do the math, you'll have to make some cunning changes of variables. One route is this: Write $\theta = \pi - 2\alpha$ and then use the relevant trig identities to replace the cosines of θ by sines of α. Now substitute $\sin\alpha = u$ and do the remaining integral.]

XC7*. Optional/extra-credit. A surface of revolution is generated as follows: Two fixed points (x_1, y_1) and (x_2, y_2) in the xy plane are joined by a curve $y = y(x)$. [Actually you'll make life easier if you start out writing this as $x = x(y)$.] The whole curve is now rotated about the x axis to generate a surface. (a) Show that the curve for which the area of the surface is stationary has the form $y = y_0 \cosh[(x - x_0)/y_0]$, where x_0 and y_0 are constants. (This is often called the soap-bubble problem, since the resulting surface is usually the shape of a soap bubble held by two coaxial rings of radii y_1 and y_2.) (b) Now consider the special case $y_1 = y_2$ where the two rings have the same radius. What is the largest value of $(x_2 - x_1)/y_1$ for which a minimal surface exists? You will need to solve something numerically here.

XC8*. Optional/extra-credit. (a) Find the Fourier coefficients a_n and b_n for the function shown in the left figure below. Make a graph similar to Figure 5.23, comparing the function itself with the first two terms in the Fourier series, and another for the first ten terms. Take $f_{\max} = 1$. (b) Repeat part (a) for the function shown in the right figure below.





XC9*. Optional/extra-credit. An oscillator is driven by the periodic force of Problem XC8, which has period $\tau = 2$. (a) Find the long-term motion $x(t)$, assuming the following parameters: natural period $\tau_0 = 2$ (that is, $\omega_0 = \pi$), damping parameter $\beta = 0.1$, and maximum drive strength $f_{\max} = 1$. Find the coefficients in the Fourier series for $x(t)$ and plot the sum of the first four terms in the series for $0 \le t \le 6$. (b) Repeat, except with natural period equal to 3.

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