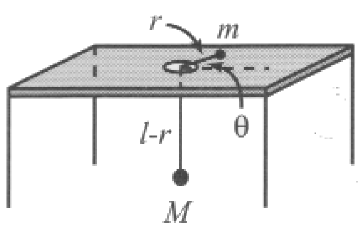
Physics 351, Spring 2015, Homework #6.

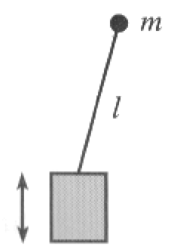
Due at start of class, Friday, February 27, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
`positron.hep.upenn.edu/q351`
to tell me how the homework went for you.

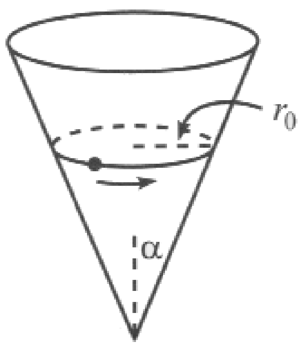
1. A mass m is free to slide on a frictionless table and is connected, via a string that passes through a hole in the table, to a mass M that hangs below. Assume that M moves in a vertical line only, and assume that the string always remains taut. (a) Find the EOM for r and for θ as shown in the left figure below. (b) Under what condition does m undergo circular motion? (c) What is the frequency of small oscillations (in the variable r) about this circular motion (i.e. if the orbit is perturbed slightly w.r.t. the circular motion)?

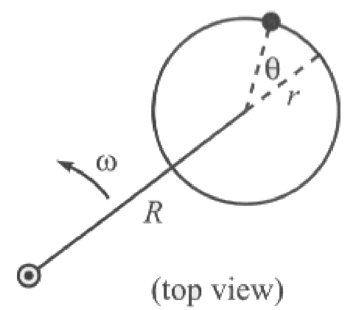




2. An "inverted pendulum" consists of a mass m at the top end of a massless stick of length l. The bottom end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$, where $A \ll l$, as shown in the above-right figure. It turns out that if ω is large enough, and if the pendulum is initially nearly upside-down, then surprisingly it will *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position. (a) Find the EOM for the angle θ of the pendulum (measured relative to the position in which the stick is perfectly vertical, with m on top). (b) By numerically integrating the EOM under suitable conditions, show that the pendulum stays up (and moves back and forth about $\theta = 0$) if $\omega > \sqrt{2gl}/A$, but falls over if $\omega < \sqrt{2gl}/A$. (I got this to work out nicely using $l = 1$, $A = 0.01$, $g = 9.8$, $\theta_0 = 0.001$.) You can just show $\theta(t)$ for a couple of points below and a couple of points above the critical frequency.

3. A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?





4. A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point, as shown in the above-right figure. (a) Find the EOM for angle θ. (b) Find the frequency of small oscillations about the point of stable equilibrium.

5. Let's modify the situation of Problem 4 so that the plane of the hoop is now vertical (so "top view" becomes "side view"), and the center of the hoop travels (in this vertical plane) in a circle of radius R with constant angular speed ω about the indicated point. (a) Find the EOM for angle θ. (b) For large ω (which implies small θ), find the amplitude of the "particular" solution with frequency ω. (c) What happens if $r = R$?

6. Consider the action, from $t = 0$ to $t = 1$, of a ball dropped from rest. From the E-L equation (or from $F = ma$), we know that $y(t) = -gt^2/2$ yields a stationary value of the action. Show explicitly that the particular function $y(t) = -gt^2/2 + \varepsilon t(t-1)$ yields an action that has no first-order dependence on ε.

7. The "spherical pendulum" is just a simple pendulum that is free to move in any sideways direction. (By contrast a "simple pendulum" — unqualified — is confined to a single vertical plane.) The bob of a spherical pendulum moves on a sphere, centered on the point of support with radius $r = R$, the length of the pendulum. A convenient choice of coordinates is spherical polars, r, θ, ϕ, with the origin at the point of support and the polar axis pointing straight down. The two variables θ and ϕ make a good choice of generalized coordinates. (a) Find the Lagrangian and the EOM for θ and for ϕ. (b) Explain what the ϕ EOM tells us about the z component of angular momentum, ℓ_z. (c) For the special case that $\phi = \text{const}$, state what familiar situation the θ EOM describes. (d) Use the ϕ EOM to eliminate $\dot{\phi}$ in favor of ℓ_z in the θ EOM and discuss the existence of an angle θ_0 at which θ can remain constant. Why is this motion called a conical pendulum? (e) Show that if $\theta = \theta_0 + \epsilon$, with $\epsilon \ll 1$, then θ oscillates about θ_0 in harmonic motion. Describe the motion of the pendulum's bob.

8. Noether's theorem asserts a connection between invariance principles and conservation laws. In §7.8 we saw that translational invariance of $\mathcal{L}$ implies conservation of total linear momentum. Here you will prove that rotational invariance of $\mathcal{L}$ implies conservation of total angular momentum. Suppose that $\mathcal{L}$ of an N-particle system is unchanged by rotations about a certain symmetry axis. (a) Without loss of generality, take this axis to be the z axis, and show that $\mathcal{L}$ is unchanged when all of the particles are simultaneously moved from $(r_\alpha, \theta_\alpha, \phi_\alpha)$ to $(r_\alpha, \theta_\alpha, \phi_\alpha + \epsilon)$ (same ϵ for all particles). Hence show that

$$\sum_{\alpha=1}^{N} \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0.$$

(b) Use Lagrange's equations to show that this implies that the total angular momentum L_z about the symmetry axis is constant. In particular, if $\mathcal{L}$ is invariant under rotations about all axes, then all components of $\vec{L}$ are conserved.

9. Let $F = F(q_1, \cdots, q_n)$ be any function of the generalized coordinates $(q_1, \cdots, q_n)$ of a system with Lagrangian $\mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$. Prove that the two Lagrangians $\mathcal{L}$ and $\mathcal{L}' = \mathcal{L} + \mathrm{d}F/\mathrm{d}t$ give exactly the same equations of motion. (This theorem turns out also to be true for the more general case $F = F(q_1, \cdots, q_n, t)$.)

10. Verify that the positions of two particles can be written in terms of the CM and relative positions as $\mathbf{r}_1 = \mathbf{R} + m_2\mathbf{r}/M$ and $\mathbf{r}_2 = \mathbf{R} - m_1\mathbf{r}/M$. Hence confirm that the total KE of the two particles can be expressed as $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$, where μ denotes the reduced mass $\mu = m_1 m_2/M$.

11. Although the main topic of Chapter 8 is the motion of two particles subject to no external forces, many of the ideas (e.g. $\mathcal{L} = \mathcal{L}_{\rm cm} + \mathcal{L}_{\rm rel}$ as in (Eq. 8.13)) extend easily to more general situations. To illustrate this, consider the following: Two masses m_1 and m_2 move in a uniform gravitational field $\mathbf{g}$ and interact via a potential energy $U(r)$. (a) Show that $\mathcal{L}$ can be decomposed as in (Eq. 8.13). (b) Write down Lagrange's equations for the three CM coordinates X, Y, Z and describe the motion of the CM. Write down the three Lagrange equations for the relative coordinates and show clearly that the motion of $\mathbf{r}$ is the same as that of a single particle of mass equal to the reduced mass μ, with position $\mathbf{r}$ and potential energy $U(r)$.

12. Two particles of masses m_1 and m_2 are joined by a massless spring of natural length L and force constant k. Initially m_2 is resting on a table and I am holding m_1 vertically above m_2 at a height L. At time $t = 0$, I project m_1 vertically upward with initial velocity v_0. Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion. Assume that v_0 is small enough that the two masses never collide.

13. Consider two particles of equal mass, $m_1 = m_2$, attached to each other by a light straight spring (force constant k, natural length L) and free to slide over a frictionless horizontal table. (a) Write down $\mathcal{L}$ in terms of the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, and rewrite it in terms of the CM and relative positions, $\mathbf{R}$ and $\mathbf{r}$, using polar coordinates (r, ϕ) for $\mathbf{r}$. (b) Write down and solve the Lagrange equations for the CM coordinates X, Y. (c) Write down the Lagrange equations for r and ϕ. Solve these for the two special cases that r remains constant and that ϕ remains constant. Describe the corresponding motions. In particular, show that the frequency of oscillations in the second case is $\omega = \sqrt{2k/m_1}$.

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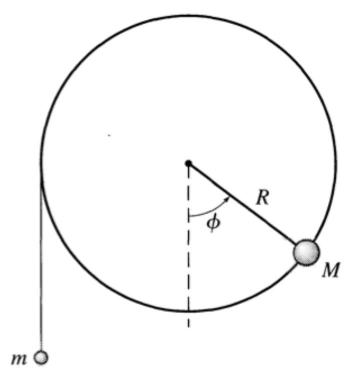
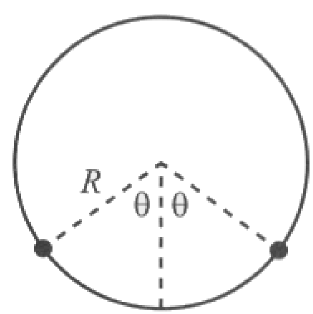
XC00*. Optional/extra-credit. If there are extra-credit problems from HW4 or HW5 that you didn't have time to do sooner, you can feel free to turn them in with HW6 for full credit. Just clearly indicate for Tanner which problem you're solving.

XC1*. Optional/extra-credit. Let the horizontal plane by the x-y plane.

A bead of mass m slides with speed v along a curve described by the function $y = f(x)$. What force does the curve apply to the bead? (Ignore gravity.)

XC2*. Optional/extra-credit. Consider a massless wheel of radius R mounted on a frictionless horizontal axis. A point mass M is glued to the edge, and a massless string is wrapped several times around the perimeter and hangs vertically down with a mass m suspended from its bottom end. (See left figure below.) Initially I am holding the wheel with M vertically below the axle. At $t = 0$, I release the wheel, and m starts to fall vertically down. (a) Write down $\mathcal{L} = T - U$ as a function of the angle ϕ through which the wheel has turned. Find the EOM and show that, provided $m < M$, there is one position of stable equilibrium. (b) Assuming $m < M$, sketch the potential energy $U(\phi)$ for $-\pi < \phi < 4\pi$ and use your graph to explain the equilibrium positions you found. (c) Because the EOM cannot be solved in terms of elementary functions, you are going to solve it numerically. This requires that you choose numerical values for the various parameters. Take $M = g = R = 1$ (this amounts to a convenient choice of units) and $m = 0.7$. Before solving the EOM, make a careful plot of $U(\phi)$ and predict the kind of motion expected when M is released from rest at $\phi = 0$. Now solve the EOM for $0 < t < 20$ and verify your prediction. (d) Repeat part (c), but with $m = 0.8$.





XC3*. Optional/extra-credit. Two equal masses are glued to a massless hoop of radius R that is free to rotate about its center in a vertical plane. The angle between the masses is 2θ, as shown in the right figure above. Find the frequency of small oscillations. (This is an easy XC problem.)

XC4*. Optional/extra-credit. Consider a particle of mass m and charge q moving in a uniform constant magnetic field $\mathbf{B}$ that points in the z direction.

(a) Prove that $\mathbf{B}$ can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ with $A = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Prove equivalently that in cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}B\rho\hat{\phi}$. (b) Write

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A})$$

in cylindrical polar coordinates and find the three corresponding Lagrange equations. (c) Describe in detail those solutions of the Lagrange equations for which ρ is a constant.

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