Physics 351, Spring 2015, Homework #7.
Due at start of class, Friday, March 6, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
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to tell me how the homework went for you.

1. Two particles whose reduced mass is μ interact via a potential energy $U = \frac{1}{2}kr^2$, where r is the distance between them. (a) Make a sketch showing $U(r)$, the centrifugal potential energy $U_{\text{cf}}(r)$, and the effective potential energy $U_{\text{eff}}(r)$. (Treat the angular momentum ℓ as a known, fixed constant.) (b) Find the "equilibrium" separation r_0, the distance at which the two particles can circle each other with constant r. (c) By Taylor-expanding $U_{\text{eff}}(r)$ to order $(r - r_0)^2$, find the frequency of small oscillations about the circular orbit if the particles are disturbed slightly from the "equilibrium" separation r_0.

2. Consider a particle of reduced mass μ orbiting in a central force with $U = kr^n$ where $kn > 0$. (a) Explain what the condition $kn > 0$ tells us about the force. Sketch the effective potential energy U_{eff} for the cases that $n = 2$, $n = -1$, and $n = -3$. (b) Find the radius r_0 at which the particle (with given angular momentum ℓ) can orbit at a fixed radius. For what values of n is this circular orbit stable? Do your sketches confirm this conclusion? (c) For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{\text{osc}} = \tau_{\text{orb}}/\sqrt{n + 2}$. Argue that if $\sqrt{n + 2}$ is a rational number, these orbits are closed. Sketch them for the cases that $n = 2$, $n = -1$, and $n = 7$.

3. We have proved in (Eq. 8.49) that any Kepler orbit can be written in the form $r(\phi) = c/(1 + \epsilon \cos \phi)$, where $c > 0$ and $\epsilon \geq 0$. For the case that $0 \leq \epsilon < 1$, rewrite this equation in rectangular coordinates (x, y) and prove that the equation can be cast in the form

$$\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a = c/(1 - \epsilon^2)$, $b = c/\sqrt{1 - \epsilon^2}$, and $d = a\epsilon$.

4. Here is a more general form of the virial theorem that applies to any periodic orbit of a particle. (a) Find the time derivative of the quantity $G = \vec{r} \cdot \vec{p}$ and,

by integrating from time 0 to t, show that

$$\frac{G(t) - G(0)}{t} = 2 \langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle$$

where $\vec{F}$ is the net force on the particle and $\langle f \rangle$ denotes the average over time of any quantity f. (b) Explain why, if the particle's orbit is periodic and we make t sufficiently large, we can make the left-hand side of this equation as small as we please. That is, the LHS$\to 0$ as $t \to \infty$. (c) Use this result to prove that if $\vec{F}$ comes from the potential energy $U = kr^n$, then $\langle T \rangle = (n/2) \langle U \rangle$, if now $\langle f \rangle$ denotes the time average over a very long time.

5. An earth satellite is observed at perigee to be 250 km above the earth's surface and traveling at about 8500 m/s. Find the eccentricity of its orbit and its height above the earth's surface at apogee. Useful data: the earth's radius is $R_e \approx 6.4 \times 10^6$ m, and $GM_e/R_e^2 = g$.

6. A particle of mass m moves with angular momentum ℓ in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

where $k > 0$ and $\lambda > 0$. (a) Write down the transformed radial equation

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2\, u(\phi)^2}\, F$$

(Eq. 8.41) and prove that the orbit has the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos(\beta\phi)}$$

where c, β, and ϵ are positive constants. (b) Find c and β in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$. (c) For what values of β is the orbit closed? What happens to your results as $\lambda \to 0$?

7. At time t_0 a comet is observed at radius r_0 traveling with speed v_0 at an acute angle α to the line from the comet to the sun. Put the sun at the origin O, with the comet on the x axis (at t_0) and its orbit in the xy plane, and then show how you could calculate the parameters of the orbital equation in the form $r = c/[1 + \epsilon \cos(\phi - \delta)]$. Do so for the case that $r_0 = 1.0 \times 10^{11}$ m, $v_0 = 45$ km/s, and $\alpha = 50°$. [The sun's mass is about 2.0×10^{30} kg, and $G = 6.67 \times 10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2}$.]

8. A particle travels in a parabolic orbit in a planet's gravitational field and skims the surface at its closest approach. The (spherical) planet has uniform mass density ρ. Relative to the center of the planet, what is the angular velocity of the particle as it skims the surface?

9. The derivation of $m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$ (Eq. 9.34), for Newton's 2nd law in a rotating frame, assumes that angular velocity $\boldsymbol{\Omega}$ is constant. Show that if $\dot{\boldsymbol{\Omega}} \neq 0$ then there is a third "fictitious force," sometimes called the *azimuthal force,* on the RHS of (9.34) equal to $m\mathbf{r} \times \dot{\boldsymbol{\Omega}}$.

10. In this problem you will prove (Eq. 9.34) using the Lagrangian approach. As usual, the Lagrangian method is in many ways easier than the Newtonian (except for some vector gymnastics), but is perhaps less insightful. Let $\mathcal{S}$ be a noninertial frame rotating with constant angular velocity $\boldsymbol{\Omega}$ relative to the inertial frame $\mathcal{S}_0$. Let both frames have the same origin, $O = O_0$. (a) Find $\mathcal{L} = T - U$ in terms of the coordinates $\mathbf{r}$ and $\dot{\mathbf{r}}$ of $\mathcal{S}$. [Remember first to evaluate T in the inertial frame. Remember also that $\mathbf{v}_0 = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}$.] (b) Show that the three Lagrange equations reproduce (9.34) precisely.

11. On a certain spherically-symmetric planet, the free-fall acceleration has magnitude $g = g_0$ at the north pole and $g = \lambda g_0$ (with $0 \leq \lambda \leq 1$) at the equator. Find $g(\theta)$, the free-fall acceleration at colatitude θ as a function of θ.

12. The center of a long frictionless rod is pivoted at the origin and the rod is forced to rotate at a constant angular velocity Ω in a horizontal plane. Write down the EOM for a bead that is threaded on the rod, using the coordinates x and y of a frame that rotates with the rod (with x along the rod and y perpendicular to it). Solve for $x(t)$. What is the role of the centrifugal force? What of the Coriolis force? (How do the normal force and the Coriolis force relate to one another?)

13. A particle of mass m is confined to move, without friction, in a vertical plane, with axes x horizontal and y vertically up. The plane is forced to rotate with constant angular velocity Ω about the y axis. Find the equations of motion for x and y, solve them, and describe the possible motions.

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XC00*. Optional/extra-credit. If there are extra-credit problems from HW4,5,6 that you didn't have time to do sooner, you can feel free to turn them in with HW7 for full credit. Just clearly indicate for Tanner which problem you're solving.

XC1*. Optional/extra-credit. Sometimes in the Calculus of Variations we want to extremize an integral subject to the constraint that another integral have a given (constant) value. Any such problem is called an *isoperimetric* problem. (See e.g. Mary Boas, 3ed, §9.6.) The original and most famous example is *Queen Dido's Problem:* of all the closed plane curves of a given perimeter, which one encloses the largest area? To solve this problem, we must maximize the area, $\int y \, dx$, subject to the condition that the arc length $\int ds$ has the given value ℓ. Let

$$I = \int_{x_1}^{x_2} F(x, y, y') \, dx$$

be the integral we want to make stationary, while the integral (with same integration variable and same limits)

$$J = \int_{x_1}^{x_2} G(x, y, y') \, dx$$

is to have a given constant value. (So the allowed varied paths must be paths for which J has the given value.) Using the Lagrange multiplier method, it can be shown that

$$\int_{x_1}^{x_2} (F(x, y, y') + \lambda G(x, y, y')) \, dx$$

should be stationary, i.e. that $F + \lambda G$ should satisfy the Euler-Lagrange equation, where the Lagrange multiplier λ is a constant. Using $F = y$ and $G = \sqrt{1 + (y')^2}$, show that the solution to Queen Dido's problem is an arc of a circle, $(x + c)^2 + (y + c')^2 = \lambda^2$, passing through the two given points.

XC2*. Optional/extra-credit. A uniform flexible chain of given length is suspended at given points (x_1, y_1) and (x_2, y_2). Using the "isoperimetric" method of Problem XC1, find the curve $y(x)$ that minimizes the chain's gravitational potential energy, subject to the constraint that its length be the given value. You will find a catenary (cosh) that looks like the solution to hw04/q8 but with a vertical offset: $y = y_0 + C \cosh((x - x_0)/C)$, which makes more physical sense than without the offset y_0.

XC3*. Optional/extra-credit. Consider a particle with mass m and angular momentum ℓ in the field of a central force $F = -k/r^{5/2}$. To simplify your equations, choose units for which $m = \ell = k = 1$. (a) Find the value r_0 at which U_{eff} is minimum and make a graph of $U_{\text{eff}}(r)$ for $0 < r \le 5r_0$. (Choose your scale so that your graph shows the interesting part of the curve.) (b) Assuming now that the particle has energy $E = -0.1$, find an accurate value of $r_{\min}$, the particle's distance of closest approach to the force center. (Use e.g. Mathematica to solve the relevant equation numerically.) (c) Assuming that the particle is at $r = r_{\min}$ when $\phi = 0$, use (e.g.) NDSolve in Mathematica to solve the transformed radial equation (Eq. 8.41)

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 \, u(\phi)^2} \, F$$

and find the orbit in the form $r = r(\phi)$ for $0 \le \phi \le 7\pi$. Graph the orbit. Does it appear to be closed?

XC4*. Optional/extra-credit. A particle moves in a potential given by $U(r) = -U_0 e^{-\lambda^2 r^2}$. (a) Given the angular momentum ℓ, find the radius of the stable circular orbit. An implicit equation is fine. (b) It turns out that if ℓ is too large, then no circular orbit exists. What is the largest value of ℓ for which a circular orbit does in fact exist? If r_0 is the radius of the circle in this cutoff case, what is the value of $U_{\text{eff}}(r_0)$? (c) To check that your answer for part (b) makes sense, make a graph of $U_{\text{eff}}(r)$ for ℓ slightly smaller than $\ell_{\max}$ and for ℓ slightly larger than $\ell_{\max}$, and interpret the condition for the existence of a stable circular orbit.

XC5*. Optional/extra-credit. A particle of mass m moves with angular momentum ℓ about a fixed force center with $F(r) = k/r^3$ where k can be positive or negative. (a) Sketch the effective potential energy U_{eff} for various values of k and describe the various possible kinds of orbit. (b) Write down and solve the transformed radial equation (8.41)

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 \, u(\phi)^2} \, F$$

and use your solutions to confirm your predictions from part (a).

XC6*. Optional/extra-credit. Consider the motion of two particles subject to a *repulsive* inverse-square force (for example, two positive charges). Show that this system has no states with $E < 0$ (as measured in the CM

frame), and that in all states with $E > 0$, the relative motion follows a hyperbola. Sketch a typical orbit. [You can follow closely the analysis of §8.6-8.7 except that you must reverse the force. Probably the simplest way to do this is to change the sign of γ in (Eq. 8.44) and all subsequent equations (so that $F(r) = +\gamma/r^2$) and then keep γ itself positive. Assume $\ell \neq 0$.]

XC7*. Optional/extra-credit. A spacecraft in a circular orbit wishes to transfer to another circular orbit of one-quarter the radius by means of a tangential thrust to move into an elliptical orbit and a second tangential thrust at the opposite end of the ellipse to move into the desired circular orbit. (The picture looks like Figure 8.13 but run backwards.) Find the thrust factors required and show that the speed in the final orbit is two times *greater* than the initial speed.
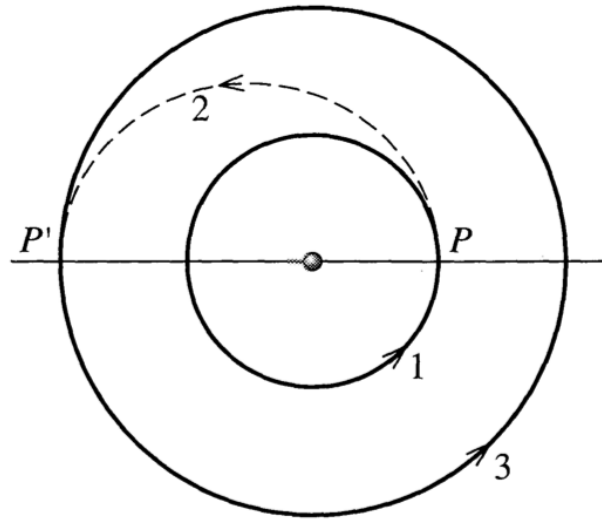

Figure 8.13   Two successive boosts, at P and P', transfer a satellite from the smaller circular orbit 1 to a transfer orbit 2 and thence to the final circular orbit 3.

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