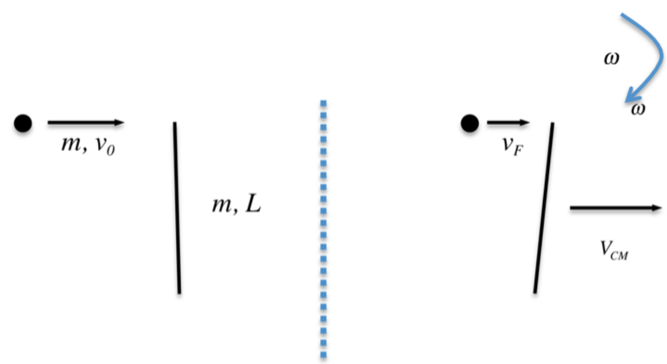
Physics 351, Spring 2015, Homework #10.

Due at start of class, Friday, April 3, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at `positron.hep.upenn.edu/q351` to tell me how the homework went for you.

1. [Here's a problem from a previous year's midterm exam.] A plank of length L and mass m lies on a frictionless plane. A ball of mass m and speed v_0 strikes the end of the plank, as shown in the figure below. The collision is *elastic.* Immediately after the collision, the ball is moving along the line of its original motion. (a) Determine the three equations that relate to conserved quantities of the motion. (b) What is v_f, the final velocity of the ball? (If you find two solutions, explain why one of them should be discarded.)

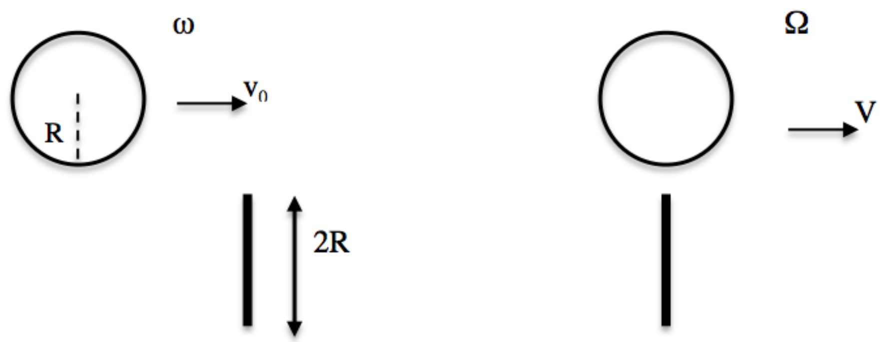


2. [Here's another problem from a previous year's midterm exam.] Consider a particle of mass m moving in a plane in a circular orbit under the influence of an attractive central Hooke's-law-like force $F(r) = -kr$, where k is a constant. The angular momentum of the particle is ℓ. (a) Find the effective potential energy of the equivalent one-dimensional problem. (b) Show that the radius R of the circular orbit is

$$R = \left(\frac{\ell^2}{mk}\right)^{1/4}$$

(c) Now consider orbits that are slightly perturbed from the circular orbit, i.e. orbits where the radius can be written $r(t) = R + \epsilon(t)$, where $\epsilon(t) \ll R$. Show that the motion is stable and calculate the **period** of the oscillations about the circular orbit.

3. ("The Lollipop.") A puck (uniform solid disk) of mass m and radius R slides across a frictionless surface, as shown in the figure below. The puck has translational velocity v_0 (pointing to the right) and is rotating with angular velocity ω (clockwise, i.e. $\boldsymbol{\omega}$ points into the page). The puck just grazes the top of a rod of mass m and length $2R$ that is initially at rest. The puck sticks to the rod (totally inelastic collision), forming a rigid body that looks like a lollipop that after the collision moves onward with center-of-mass velocity V and angular velocity Ω. (a) If the puck's initial translational velocity and its initial rotational velocity are related by $v_0 = \omega R$, what is the resulting angular speed Ω of the lollipop? (b) How much energy is lost (dissipated) in the collision?



4. Consider a rigid plane body (a "lamina"), such as a flat piece of sheet metal, rotating about a point O on the body. If we choose axes so that the lamina lies in the xy plane, which elements of the inertia tensor $\boldsymbol{I}$ are automatically zero? Prove that $I_{zz} = I_{xx} + I_{yy}$.

5. A thin, flat, uniform metal triangle lies in the xy plane with its corners at $(1, 0, 0)$, $(0, 1, 0)$, and the origin. Its surface density (mass/area) is $\sigma = 24$. (Let's measure distances and masses in unspecified units, with the number 24 chosen to make the answer come out nicely.) (a) Find the triangle's inertia tensor $\boldsymbol{I}$. (b) What are its principal moments and the corresponding axes? [Since the object is flat, the z axis is automatically a principal axis. (Do you see why?) As you proved in problem 4, $I_{zz} = I_{xx} + I_{yy}$ for a flat object in the xy plane.]

6. (a) If $\boldsymbol{I}^{\text{cm}}$ denotes the moment-of-inertia tensor of a rigid body (mass M) about its CM, and $\boldsymbol{I}$ the corresponding tensor about a point P displaced from the CM by $\boldsymbol{\Delta} = (\xi, \eta, \zeta)$, prove that $I_{xx} = I^{\text{cm}}_{xx} + M\,(\eta^2 + \zeta^2)$, that

$I_{yz} = I_{yz}^{\mathrm{cm}} - M\eta\zeta$, and so forth. (Prove these first two statements, then simply write down the "and so forth" results by cyclic permutation.) These results, which generalize the familiar parallel-axis theorem, mean that once you know the inertia tensor about the CM, it is easily calculated for any other origin. (b) Use your results to confirm the results of Example 10.2 (page 381). In other words, given that $I_{xx} = I_{yy} = I_{zz} = Ma^2/6$ (and all off-diagonal elements zero) for a cube about its center (with edges parallel to the xyz axes), use your results to confirm that for the same cube about its corner (with edges still parallel to the xyz axes), the inertia tensor has diagonal elements $\frac{2}{3}Ma^2$ and off-diagonal elements $-Ma^2/4$.

7. (a) Find all nine elements of the inertia tensor w.r.t. the CM of a uniform cuboid (a rectangular brick shape) whose sides are $2a$, $2b$, and $2c$ in the x, y, z directions, and whose mass is M. Explain clearly why you could write down the off-diagonal elements without doing any integration. (b) Combine the result of part (a) and problem 6 to find the inertia tensor of the same cuboid w.r.t. the corner A at (a, b, c). (c) What is the angular momentum about A if the cuboid is spinning with angular velocity ω around the edge through A and parallel to the x axis?

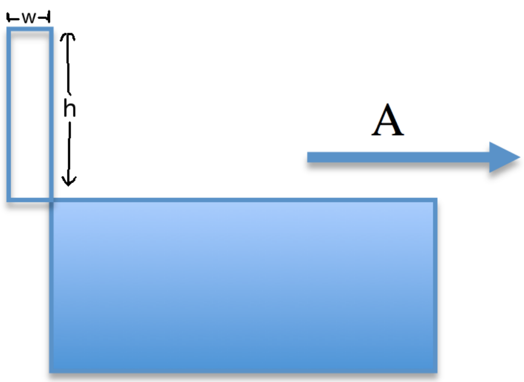
8. A frictionless hoop of radius R is made to rotate at constant angular speed ω around a diameter. A bead on the hoop starts on this diameter (i.e. where the diameter meets the hoop) and is then given a tiny kick. Let $\boldsymbol{N}$ be the total force that the hoop exerts on the bead, and let $N_\perp$ be the component of $\boldsymbol{N}$ that is perpendicular to the plane of the hoop. Where is $N_\perp$ maximum? What is the magnitude of $\boldsymbol{N}$ as a function of position? (Ignore gravity in this problem.)

9. A coin stands upright at an arbitrary point on a rotating turntable (constant $\boldsymbol{\Omega}$), and spins (without slipping) at the required angular speed to make the coin's center remain motionless in the lab frame. (Therefore what is the magnitude of the frictional force acting on the coin?) In the frame of the turntable, the coin rolls around in a circle with the same frequency as that of the turntable (in what direction?). Now analyze the forces and torques acting on the coin from the perspective of a camera mounted on the turntable. Working in the frame of the turntable (thus accounting for pseudoforces (and maybe also "pseudotorques?") in $\sum \boldsymbol{F}$ and $\sum \boldsymbol{\tau}$), show that

(a) $\sum \boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, and

(b) $\sum \boldsymbol{\tau} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$ (Hint: Coriolis (!) — see problems XC3 and XC4)

10. (Accelerating reference frames.) A truck is at rest with its rear door fully open. The truck then accelerates with constant acceleration A, and the door swings shut. The door is uniform and solid, with mass M, "height" h (really the door's width), and "width" w (the door's thickness), as shown in the left figure below (which is a top view). Ignore air resistance and friction. (a) Find the instantaneous angular velocity of the door about its hinges when the door has swung through 90°. (b) Find the horizontal (i.e. the component parallel to $\boldsymbol{A}$) force on the door when the door has swung through 90°. (Find the real force acting on the door, not counting the inertial pseudoforce.) [It is best to work this problem in the (accelerating) frame of the truck. The inertial force $-M\boldsymbol{A}$ can be thought of as acting on the CM of the door.]



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XC00*. Optional/extra-credit. If there are XC problems from earlier homeworks that you didn't have time to do sooner, you can still turn them in for full credit. Just clearly indicate for Tanner which problem you're solving.

XC1. Optional/extra-credit. Find the inertia tensor for a uniform, thin hollow cone, such as an ice-cream cone, of mass M, height h, and base radius R, spinning about its pointed end.

XC2. Optional/extra-credit. (a) A triangular prism (like a box of Toblerone) of mass M, whose two ends are equilateral triangles parallel to the xy plane with side $2a$, is centred on the origin with its axis along the z axis. Find its moment of inertia for rotation about the z axis. Without doing any integrals, write down and explain its two products of inertia for rotation about the z axis. (b) Find the inertia tensor $\boldsymbol{I}$ for the triangular prism of part (a), with height h. (You've already done about half the work in part (a).) Your result should show that $\boldsymbol{I}$ has the form we've found for an axisymmetric body. This suggests what is true, that three-fold symmetry about an axis (symmetry under rotations of $120°$) is enough to ensure this form.

XC3. Optional/extra-credit. The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r spinning with angular velocity ω about its vertical axis at colatitude θ. Show that the Coriolis force due to the earth's rotation produces a torque of magnitude $m\omega\Omega r^2 \sin\theta$ directed to the west, where Ω is the earth's angular velocity. This torque is the basis of the gyrocompass.

XC4. Optional/extra-credit. The **Compton generator** is a beautiful demonstration of the Coriolis force due to the earth's rotation, invented by the American physicist A.H. Compton (1892-1962, best known as author of the Compton effect) while he was still an undergraduate. A narrow glass tube in the shape of a torus or ring (radius R of the ring $\gg$ radius of the tube) is filled with water, plus some dust particles to let one see any motion of the water. The ring and water are initially stationary and horizontal, but the ring is then spun through $180°$ about its east-west diameter. Explain why this should cause the water to move around the tube. Show that the speed of the water just after the $180°$ turn should be $2\Omega R\cos\theta$, where Ω is the earth's angular velocity, and θ is the colatitude of the experiment. What would this speed be if $R \approx 1$ m and $\theta = 40°$? Compton measured this speed with a microscope and got agreement within 3%.

XC5. Optional/extra-credit. At a point P on the earth's surface, an enormous perfectly flat and frictionless platform is built. The platform is exactly horizontal — that is, perpendicular to point P's local free-fall acceleration $\boldsymbol{g}_P$. Find the EOM for a puck sliding on the platform and show that it has the same form as (Eq. 9.61) for the Foucault pendulum

$$\ddot{x} = -gx/L \,+\, 2\dot{y}\Omega\cos\theta$$
$$\ddot{y} = -gy/L \,-\, 2\dot{x}\Omega\cos\theta$$

except that the pendulum's length L is replaced by the earth's radius R. What is the frequency of the puck's oscillations and what is that of its Foucault precession? [Hints: Write the puck's position vector, relative to the earth's center O, as $\boldsymbol{R} + \boldsymbol{r}$, where $\boldsymbol{R}$ is the position of the point P and $\boldsymbol{r} = (x, y, 0)$ is the puck's position relative to P. The contribution to the centrifugal force involving $\boldsymbol{R}$ can be absorbed into $\boldsymbol{g}_P$, and the contribution involving $\boldsymbol{r}$ is negligible. The restoring force comes from the variation of $\boldsymbol{g}$ as the puck moves.] To check the validity of your approximations, compare the approximate size of the gravitational restoring force, the Coriolis force, and the neglected term $m(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega}$ in the centrifugal force.

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