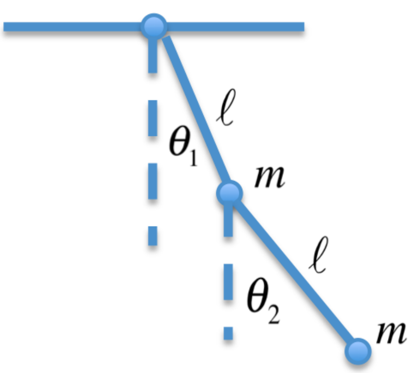
Physics 351, Spring 2015, Homework #11.

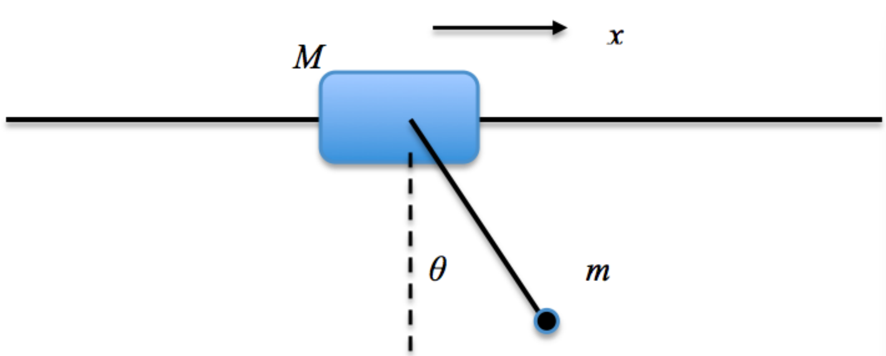
Due at start of class, Friday, April 10, 2015

Course info is at positron.hep.upenn.edu/p351

When you finish this homework, remember to visit the feedback page at positron.hep.upenn.edu/q351 to tell me how the homework went for you.

1. [Here's a problem that appeared on a previous year's midterm exam; the last two parts were extra-credit on the exam.] Consider the double pendulum consisting of two bobs confined to move in a plane. The rods are of equal length ℓ, and the bobs have equal mass m. The generalized coordinates used to describe the system are θ_1 and θ_2, the angles that the rods make with the vertical (see left figure below). (a) Write the Lagrangian for the system. (This could be an opportunity to practice writing $(\boldsymbol{v}_1 + \boldsymbol{v}_2)^2 = v_1^2 + v_2^2 + 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$.) (b) Next, simplify your Lagrangian from part (a) by assuming that angles θ_1 and θ_2 are both small. Keep terms up to second order in the angles, the angular velocities, and their products. (c) Find the two Lagrange equations of motion, which will be a set of coupled, linear differential equations. (d) Solve the equations of motion (e.g. using the techniques of Chapter 11).





2. [Here's another problem from a previous year's midterm exam; the last part was extra-credit on the exam.] A block of mass M moves on a frictionless horizontal rail. A pendulum of length L and mass m hangs from the block. (See right figure above.) Let x be the displacement of the block, and let θ be the angular displacement of the pendulum w.r.t. the vertical. (a) Write the Lagrangian for the system. (Another possible opportunity to write $(\boldsymbol{v}_1 + \boldsymbol{v}_2)^2 = v_1^2 + v_2^2 + 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$?) (b) Which of the coordinates is ignorable ("cyclic")? What is the associated conserved quantity? This is an example of what conservation law? (c) Find the Lagrange equations of motion for the system. (d) Simplify the equations of motion found in part (c) for the case of small oscillations (where you can discard any terms of second order or higher in the displacements, velocities, or their products). (e) Solve the system of differential equations from part (d) and determine the most general motion of the system. (Your solution should have four arbitrary constants. Using the results of part (b) should help you to simplify the problem.)

3. A rigid body consists of three equal masses fastened at the positions $(a, 0, 0)$, $(0, a, 2a)$, $(0, 2a, a)$. (a) Find the inertia tensor $\underline{\underline{\boldsymbol{I}}}$. (b) Find the principal moments and a set of orthogonal principal axes. (If you don't feel like doing it by hand, just use Wolfram Alpha, or Mathematica, etc.) (c) For this inertia tensor, is the choice of principal axes unique? Why or why not? If not, what linear combinations of your previously found principal axes would also be principal axes?

4. (a) A rigid body is rotating freely, subject to zero torque. Use Euler's equations (Eq. 10.88) to prove that the magnitude of the angular momentum $\boldsymbol{L}$ is constant. (Multiply the ith equation by $L_i = \lambda_i \omega_i$ and add the three equations.) (b) In much the same way, show that the kinetic energy of rotation $T_{\text{rot}} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$, as in (Eq. 10.68), is constant.

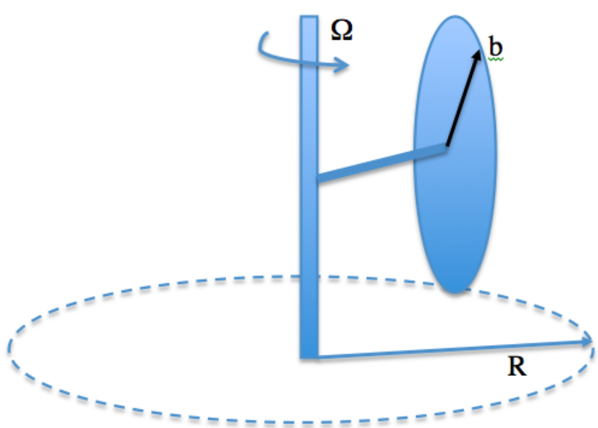
5. We saw in §10.8 that in the free precession of an axially symmetric body ($\lambda_1 = \lambda_2$) the three vectors $\hat{\boldsymbol{e}}_3$ (the symmetry axis), $\boldsymbol{\omega}$, and $\boldsymbol{L}$ lie in a plane. As seen in the body frame, $\hat{\boldsymbol{e}}_3$ is fixed, and $\boldsymbol{\omega}$ and $\boldsymbol{L}$ precess around $\hat{\boldsymbol{e}}_3$ with angular velocity $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1$. As seen in the space frame, $\boldsymbol{L}$ is fixed, and $\boldsymbol{\omega}$ and $\hat{\boldsymbol{e}}_3$ precess around $\boldsymbol{L}$ with angular frequency Ω_s. In this problem you will find three equivalent expressions for Ω_s. (a) Argue that $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. [Remember that relative angular velocities add like vectors.] (b) Bearing in mind that $\boldsymbol{\Omega}_b$ is parallel to $\hat{\boldsymbol{e}}_3$, prove that $\Omega_s = \omega \sin\alpha / \sin\theta$, where α is the angle between $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{\omega}$, and θ is the angle between $\hat{\boldsymbol{e}}_3$ and $\boldsymbol{L}$. (See Figure 10.9.) (c) Thence prove that

$$\Omega_s = \omega \frac{\sin\alpha}{\sin\theta} = \frac{L}{\lambda_1} = \omega\, \frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}$$

6. Consider the rapid steady precession of a symmetric top predicted in connection with (Eq. 10.112). (a) Show that in this motion the angular momentum $\boldsymbol{L}$ must be very close to the vertical. [Hint: Use (Eq. 10.100) to write down the horizontal component $L_{\rm hor}$ of $\boldsymbol{L}$. Show that if $\dot\phi$ is given by the right side of (Eq. 10.112), $L_{\rm hor}$ is exactly zero.] (b) Use this result to show that the rate of precession Ω given in (Eq. 10.112) agrees with the free precession rate Ω_s found in (Eq. 10.96).

7. In the discussion of steady precession of a top in §10.10, the rates Ω at which steady precession can occur were determined by the quadratic equation (Eq. 10.110). In particular, we examined this equation for the case that ω_3 is very large. In this case you can write the equation as $a\Omega^2 + b\Omega + c = 0$ where b is very large. (a) Verify that when b is very large, the two solutions of this equation are approximately $-c/b$ (which is small) and $-b/a$ (which is large). What precisely does the condition "b is very large" entail? (You should find a dimensionless ratio $\gg 1$.) (b) Verify that these give the two solutions claimed in (Eq. 10.111) and (Eq. 10.112).

8. [Here's a problem from last year's final exam.] In a "rolling mill," grain is ground by a disk-shaped millstone that rolls in a circle on a flat surface and is driven by a vertical shaft. Assume that the millstone is a uniform disk of radius b and negligible thickness. ($\lambda_3 = \frac{1}{2}mb^2$. What is $\lambda_1 = \lambda_2$, given that this object is planar?) Also assume that the wheel cannot tip, so it always remains perpendicular to the ground. The wheel rolls without slipping along a circle of radius R with angular velocity Ω as indicated in the figure. Show that the normal force that the ground exerts on the wheel is $Mg + \frac{1}{2}Mb\Omega^2$. Because of the angular momentum of the millstone, the contact force with the surface can be much larger than the weight of the wheel, which is what makes this an effective way to grind grain.



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XC00*. Optional/extra-credit. If there are extra-credit problems from earlier homeworks that you didn't have time to do sooner, you can feel free to turn them in with HW11 for full credit. Just clearly indicate for Tanner which problem you're solving.

XC1. Optional/extra-credit. An important special case of the motion of a symmetric top occurs when it spins about a vertical axis. Analyze this motion as follows: (a) By inspecting the effective PE (Eq. 10.114), show that if at any time $\theta = 0$, then L_3 and L_z must be equal. (b) Set $L_z = L_3 = \lambda_3\omega_3$ and then make a Taylor expansion of $U_{\text{eff}}(\theta)$ about $\theta = 0$ to terms of order θ^2. (c) Show that if $\omega_3 > \omega_{\text{min}} = 2\sqrt{MgR\lambda_1/\lambda_3^2}$, then the position $\theta = 0$ is stable, but if $\omega_3 < \omega_{\text{min}}$ it is unstable. (In practice, friction slows the top's spinning. Thus with ω_3 sufficiently fast, the vertical top is stable, but as it slows down the top will eventually lurch away from the vertical when ω_3 reaches ω_{min}.)

XC2. Optional/extra-credit. [Computer] The nutation of a top is controlled by the effective potential energy (Eq. 10.114). Make a graph of $U_{\text{eff}}(\theta)$ as follows: (a) First, since the second term of $U_{\text{eff}}(\theta)$ is a constant, you can ignore it. Next, by choice of your units, you can take $MgR = 1 = \lambda_1$. The remaining parameters L_z and L_3 are genuinely independent parameters. To be definite set $L_z = 10$ and $L_3 = 8$ and plot $U_{\text{eff}}(\theta)$ as a function of θ. (b) Explain clearly how you would use your graph to determine the angle θ_0 at which the top could precess steadily with $\theta =$ constant. Find θ_0 to three significant figures. (c) Find the rate of this steady precession, $\Omega = \dot{\phi}$, as given by (Eq. 10.115). Compare with the approximate value of Ω given by (Eq. 10.112).

XC3. Optional/extra-credit. Do Taylor's problem 10.33 (page 412), which is too long to retype here. It involves deriving expressions for T and for $\boldsymbol{L}$.

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