Physics 351, Spring 2015, Homework #12.

Due at start of class, Friday, April 17, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
`positron.hep.upenn.edu/q351`
to tell me how the homework went for you.

1. *Hamiltonian treatment of the symmetric top.* [Here's a problem that appeared on a previous year's final exam.] Consider a symmetric top ($\lambda_1 = \lambda_2$) whose tip has a fixed location in space. Using the Euler angles ϕ, θ, and ψ (whose detailed definitions are not needed for you to solve this problem) to represent the top's orientation, the top's Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\lambda_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

where M is the mass of the top and R is the distance from the contact point to the top's CoM. λ_3 is the moment of inertia for the top's symmetry axis, and λ_1 is the moment of inertia for the other two principal axes. (a) Calculate the three generalized momenta, p_ϕ, p_θ, and p_ψ. (b) The simplest way to construct the Hamiltonian is to realize that the coordinates are natural, so $H = T + U$. Use this to show that the Hamiltonian is given by

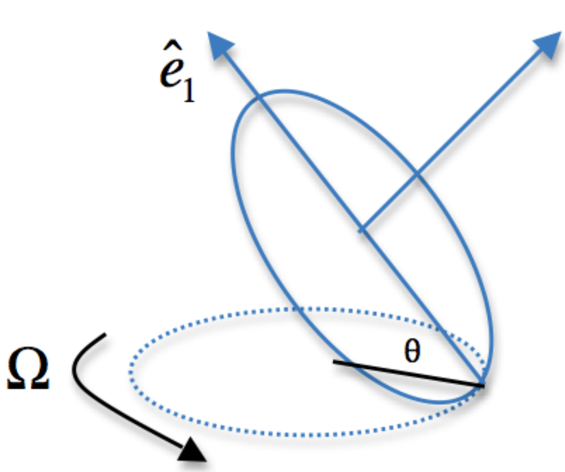
$$H = \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3} + MgR\cos\theta$$

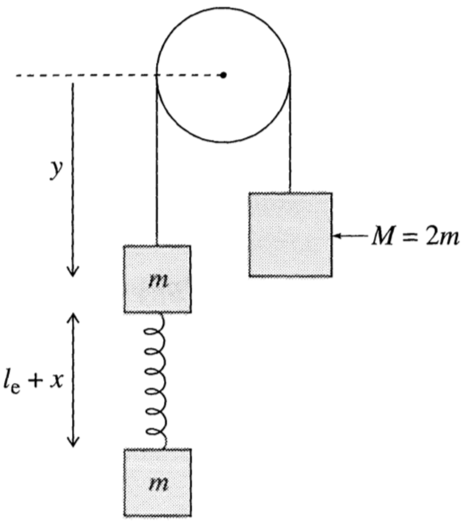
(c) Two of the Euler-angle coordinates are ignorable. Which ones? The corresponding generalized momenta are constant. Use this to show that the Hamiltonian can be written as

$$H = \frac{p_\theta^2}{2\lambda_1} + U_{\text{eff}}(\theta)$$

What is the effective potential energy U_{eff} for this system?

2. When you spin a coin around a vertical diameter on a table, it will lose energy and go into a wobbling motion, whose frequency increases as the coin's angle w.r.t. horizontal decreases. Consider the moment when the coin makes an angle θ w.r.t. the horizontal surface of the table. Assume that the CM of the coin is motionless and that the contact point moves along a circle on the table, as shown in the left figure below. Let the radius of the coin be R, and let Ω be the angular velocity of the motion of the contact point. Assume that the coin rolls without slipping. (a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega \sin\theta \, \hat{\boldsymbol{e}}_1$, where $\hat{\boldsymbol{e}}_1$ points upward along the coin, diametrically away from the contact point. (b) Show that $\Omega = 2\sqrt{g/(R\sin\theta)}$.





3. Consider the modified Atwood machine shown in the right figure above. The two weights on the left have equal masses m and are connected by a massless spring of Hooke's-law constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$, where l_e is the equilibrium length (with all the weights in position and M held stationary). (a) Show that the total potential energy (spring plus gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that we can take to be zero). (b) Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$, and write down the Hamiltonian. Show that the coordinate y is ignorable. (c) Write down the four Hamilton equations an solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m down a distance x_0, and at $t = 0$ you let go of both masses. [Hint: Write down the initial values of x, y, and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.] Describe the motion. In particular, find the frequency with which x oscillates.

4. Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw there that there have to be maximum and minimum heights $z_{\max}$ and $z_{\min}$, beyond which the mass cannot stray. When z is a maximum or minimum, it must be that $\dot{z} = 0$. Show that this can happen if and only if the conjugate momentum $p_z = 0$, and use the equation $\mathcal{H} = E$, where $\mathcal{H}$ is the Hamiltonian function (Eq. 13.3), to show that, for a given energy E, this occurs at exactly two values of z. [Hint: Write down the function $\mathcal{H}$ for the case that $p_z = 0$ and sketch its behavior as a function of z for $0 < z < \infty$. How many times can this function equal any given E?] Use your sketch to describe the motion of the mass.

5. Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw that there are solutions for which the mass remains at the fixed height $z = z_0$, with fixed angular velocity $\dot{\phi}_0$ say. (a) For any chosen value of p_ϕ, use (Eq. 13.34) to get an equation that gives the corresponding value of the height z_0. (b) Use the equations of motion to show that this motion is stable. That is, show that if the orbit has $z = z_0 + \epsilon$ with ϵ small, then ϵ will oscillate about zero. (c) Show that the angular frequency of these oscillations is $\omega = \sqrt{3}\,\dot{\phi}_0 \sin\alpha$, where α is the half angle of the cone ($\tan\alpha = c$ where c is the constant in $\rho = cz$). (d) Find the angle α for which the frequency of oscillation ω is equal to the orbital angular velocity $\dot{\phi}_0$, and describe the motion for this case.

6. All of the examples in Taylor's Chapter 13 and all of the problems (except this one) treat forces that come from a potential energy $U(\boldsymbol{r})$ [or occasionally $U(\boldsymbol{r}, t)$]. However, the proof of Hamilton's equations given in §13.3 applies to any system for which Lagrange's equations hold, and this can include forces not derivable from a potential energy. An important example of such a force is the magnetic force on a charged particle. (a) Use the Lagrangian (Eq. 7.103) to show that the Hamiltonian for a charge q in an electromagnetic field is $\mathcal{H} = (\boldsymbol{p} - q\boldsymbol{A})^2/(2m) + qV$. (This Hamiltonian plays an important role in the quantum mechanics of charged particles.) (b) Show that Hamilton's equations are equivalent to the familiar Lorentz force equation $m\ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

7. Two masses m_1 and m_2 are joined by a massless spring (force constant k and natural length l_0) and are confined to move in a frictionless horizontal plane, with CM and relative positions $\boldsymbol{R}$ and $\boldsymbol{r}$ as defined in §8.2. (a) Write down the Hamiltonian $\mathcal{H}$ using as generalized coordinates X, Y, r, ϕ, where (X, Y) are the rectangular components of $\boldsymbol{R}$, and (r, ϕ) are the polar coordinates of

$\boldsymbol{r}$. Which coordinates are ignorable and which are not? Explain. (b) Write down the 8 Hamilton equations of motion. (c) Solve the r equations for the special case that $p_\phi = 0$ and describe the motion. (d) Describe the motion for the case that $p_\phi \neq 0$ and explain physically why the r equation is harder to solve in this case.

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XC00*. Optional/extra-credit. If there are extra-credit problems from earlier homework assignments that you didn't have time to do sooner, you can feel free to turn them in with HW12 for full credit. Just clearly indicate for Tanner which problem you're solving.

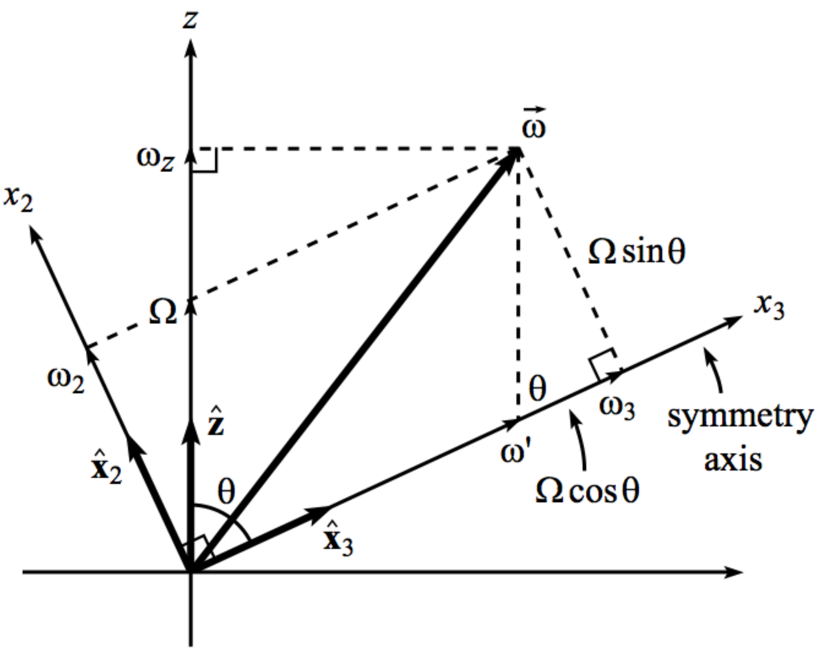
XC1. Optional/extra-credit. Consider a rotating reference frame such as a frame fixed on the earth's surface. A particle is thrown vertically up with initial speed v_0, reaches a maximum height, and falls back to the ground. Show that the Coriolis deflection when it reaches the ground is four times as large as and in the opposite direction from the Coriolis deflection when it is dropped from rest at the same maximum height. Can you explain why?

XC2. Optional/extra-credit. Assume that a piece of toast is a rigid uniform square of side length ℓ. You butter the toast and then drop it from a height H above a table; the table is a height h above the floor. The toast starts off parallel to the table, and as it falls, it clips the edge of the table and collides elastically, causing the toast to start to rotate. You want to find the value of H, in terms of h and ℓ, that leads to the sad situation in which the toast makes exactly one-half revolution and lands on the floor butter-side-down. Show that $H = \frac{\pi^2 \ell^2}{6\,(6h - \pi\ell)}$. [Hint: with a clever choice of origin, you can argue that the angular momentum of the toast is conserved during the collision with the table.]

XC3. Optional/extra-credit. (a) A small ball of radius r and uniform density rolls without slipping at the bottom of a fixed cylinder of radius $R \gg r$. Show that the frequency of small oscillations is $\omega = \sqrt{\frac{5g}{7R}}$. [You'll need $I = \frac{2}{5}Mr^2$ for a uniform sphere about an axis through its center.] (b) Generalize your result for the case where the sphere's density is not uniform (but is still spherically symmetric), so its moment of inertia is given by $I = \beta M r^2$.

XC4. Optional/extra-credit. Consider a top made of a wheel with all its mass on the rim. A massless rod (perpendicular to the plane of the wheel) connects the CM to a pivot. Initial conditions have been set up so that the top undergoes precession, with the rod always horizontal. In the language of the figure below (Morin's Fig. 9.30), we may write the angular velocity of the top as $\boldsymbol{\omega} = \Omega\hat{\boldsymbol{z}} + \omega'\hat{\boldsymbol{x}}_3$ (where $\hat{\boldsymbol{x}}_3 = \hat{\boldsymbol{e}}_3$ is horizontal here). Consider things in the frame rotating around the $\hat{\boldsymbol{z}}$ axis with angular speed Ω. In this frame, the top spins with angular speed ω' around its *fixed* symmetry axis. Therefore, in this frame we must have $\boldsymbol{\tau} = 0$, because $\boldsymbol{L}$ is constant. Verify explicitly that

$\boldsymbol{\tau} = 0$ (calculated w.r.t. the pivot) in this rotating frame (you will need to find the relation between ω' and Ω). In other words, show that the torque due to gravity is exactly canceled by the torque due to the Coriolis force (you can quickly show that the centrifugal force provides no net torque). Remember that HW10/XC3 implies a Coriolis torque of magnitude $m\omega'\Omega r^2$.



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