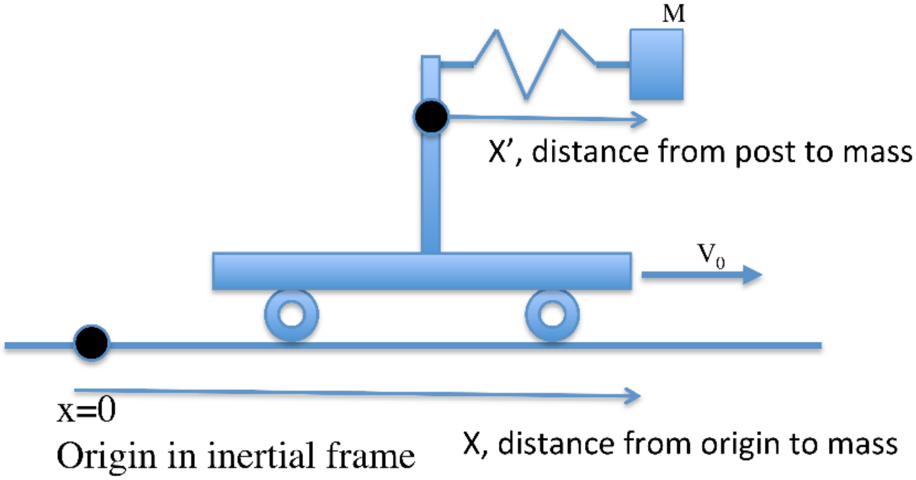
Physics 351, Spring 2015, Homework #13.

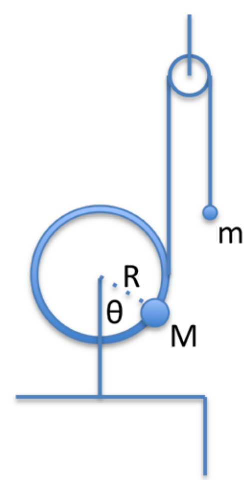
Due at start of class, Friday, April 24, 2015

Course info is at `positron.hep.upenn.edu/p351`

When you finish this homework, remember to visit the feedback page at
`positron.hep.upenn.edu/q351`
to tell me how the homework went for you.

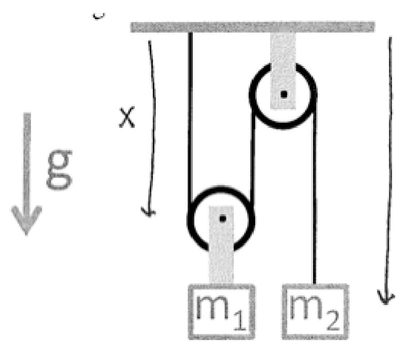
1. Consider a point mass M attached to a spring (force constant k), whose other end is attached to a massless cart that is moved by an external device at the constant speed v_0. You will consider $\mathcal{H}$ for the system using two different generalized coordinates. (a) First, consider the system using the variable x, which is referenced to a fixed origin. (See left figure below.) Write down $\mathcal{L}$ and find the Lagrange EOM for x. Now construct $\mathcal{H}$. Does $\mathcal{H}$ equal the total energy of the system? Is $\mathcal{H}$ conserved? Explain why your answers are OK. (b) Second, analyze the system again using the "relative coordinate" x', which is the displacement of the point mass relative to the cart. (x' is measured from the equilibrium position of the mass M.) Write $\mathcal{L}$ and find the Lagrange EOM for x'. Construct $\mathcal{H}$. ($\mathcal{H}$ is different from what you found in the first part because you are using a different coordinate.) Is $\mathcal{H}$ the total energy of the system? Is $\mathcal{H}$ conserved? Explain why your answers are OK.

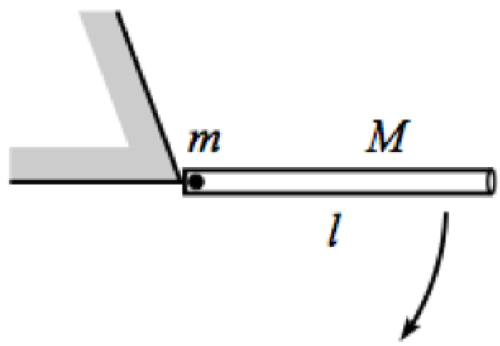




2. [This problem is from a past year's exam.] A mass M is attached to a massless hoop of radius R that lies in a vertical plane and is free to rotate about its fixed center. M is tied to a string that winds part way around the hoop and then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string. (See above-right figure.) Find the EOM for the angle θ of rotation of the hoop, where $\theta = 0$ would put M directly below the center of the hoop. What is the frequency of small oscillations about the equilibrium angle θ_0? Assume that m moves only vertically and that $M > m$.

3. Here is a problem that is a weird variation of an Atwood's machine. The goal is to find the accelerations of m_1 and m_2, as shown in the left figure below. Also find the tension in the string! Assume that the pulleys are massless and frictionless, so that the tension in the string is constant. The preferred way to solve this problem is in the Lagrangian framework, using a Lagrange multiplier; the constraint can be expressed at $2x + y = L$, the total length of the string.





4. A tube of mass M and length ℓ is free to swing by a pivot at one end. (Use the moment of inertia of a uniform thin rod rotating about one end.) A mass m is positioned inside the tube at this end. The tube is held horizontal and then released. (See above-right figure.) Let θ be the angle of the tube w.r.t. the horizontal, and let x be the distance the mass has traveled along the tube. Find the Lagrange equations of motion for θ and x, then write them in terms of θ and $\eta \equiv x/\ell$ (the fraction of the distance along the tube). These equations can only be solved numerically, and you must pick a numerical value for the ratio $r \equiv m/M$ in order to do this. Use Mathematica (or your favorite alternative) to find the value of η when the tube is vertical ($\theta = \pi/2$). Give this value of η for a few values of r.

5. Consider a function $f(q, p)$ of the coordinates q and p. Use Hamilton's equations to show that the time derivative of f can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q}$$

Note: this combination of partial deriviates comes up often enough to warrant a name. The *Poisson bracket* of two functions, f_1 and f_2, is defined to be

$$\{f_1, f_2\} \equiv \frac{\partial f_1}{\partial q}\frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial q}$$

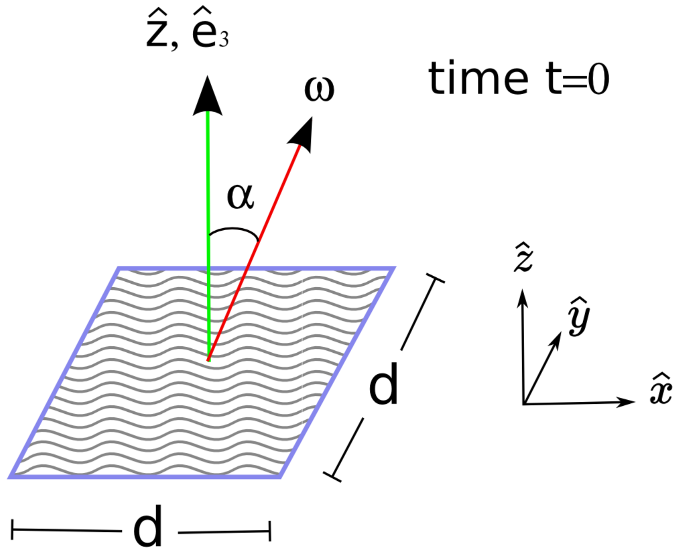
With this definition, the time derivative of f takes the nice compact form, $\mathrm{d}f/\mathrm{d}t = \{f, \mathcal{H}\}$. (More generally, for $f(q,p,t)$, $\mathrm{d}f/\mathrm{d}t = \{f, \mathcal{H}\} + \partial f/\partial t$.) Once you've seen *commutators* in quantum mechanics, you may enjoy this analogy: $[A,B] = i\hbar\,\{A,B\}$. To give just one of many examples, the classical result $\{x, p_x\} = 1$ has the QM analogy $[x, p_x] = i\hbar$.

6. Consider the chaotic motion of a DDP for which the Liapunov exponent is $\lambda = 1$, with time measured in units of the drive period as usual. (a) Suppose that you need to predict $\phi(t)$ with an accuracy of 0.01 radian and that you know the initial value $\phi(0)$ within 10^{-6} rad. What is the maximum time $t_{\max}$ for which you can predict $\phi(t)$ within the required accuracy? This $t_{\max}$ is sometimes called the **time horizon** for prediction within a specified accuracy. (b) Suppose that, with a vast expenditure of money and labor, you manage to improve the accuracy for your initial value to 10^{-9} radians (a 1000× improvement). What is the time horizon now (for the same required accuracy of prediction)? By what factor has $t_{\max}$ improved? Your results illustrate the difficulty of making accurate long-term predictions for chaotic motion.

7. A beam of particles is moving along a particle accelerator's "beam pipe" in the z direction. The particles are uniformly distributed in a cylindrical volume of length L_0 (in the z direction) and radius R_0. The particles have momenta uniformly distributed with p_z in an interval $p_0 \pm \Delta p_z$ and the transverse momentum $p_\perp$ inside a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing does not affect either L_0 or Δp_z.)

8. [This problem is adapted from a problem on the final exam I took for the analogous course in fall 1990.] A uniform, infinitesimally thick, square plate of mass m and side length d is allowed to undergo torque-free rotation. At time $t = 0$, the normal to the plate, $\hat{\boldsymbol{e}}_3$, is aligned with $\hat{\boldsymbol{z}}$, but the angular velocity vector $\boldsymbol{\omega}$ deviates from $\hat{\boldsymbol{z}}$ by a small angle α. The figure below depicts the situation at time $t = 0$, at which time $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{z}}$, and $\boldsymbol{\omega} = \omega(\cos\alpha\,\hat{\boldsymbol{z}} + \sin\alpha\,\hat{\boldsymbol{x}})$.

(a) Show that the inertia tensor has the form $\underline{\underline{\boldsymbol{I}}} = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and find the constant I_0. (b) Calculate the angular momentum vector $\boldsymbol{L}$ at $t = 0$. (c) Draw a sketch showing the vectors $\hat{\boldsymbol{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{L}$ at $t = 0$. Be sure that the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ makes sense. This relative orientation is different for frisbee-like ("oblate") objects ($\lambda_3 > \lambda_1$) than it is for the American-football-like ("prolate") object ($\lambda_3 < \lambda_1$) drawn on Taylor's page 400. (d) Draw and label the "body cone" and the "space cone" on your sketch. (e) Calculate the precession frequencies Ω_{body} and Ω_{space}. Indicate the directions of the precession vectors $\boldsymbol{\Omega}_{\mathbf{body}}$ and $\boldsymbol{\Omega}_{\mathbf{space}}$ on your drawing. (You puzzled through these directions when you solved problem 10.46.) (f) You argued in HW11 that $\boldsymbol{\Omega}_{\mathbf{space}} = \boldsymbol{\Omega}_{\mathbf{body}} + \boldsymbol{\omega}$. Verify (by writing out components) that this relationship holds for the $\boldsymbol{\Omega}_{\mathbf{space}}$ and $\boldsymbol{\Omega}_{\mathbf{body}}$ that you calculate for $t = 0$. (g) Find the maximum angle between $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{e}}_3$ during subsequent motion of the plate. Show that in the limit $\alpha \ll 1$, this maximum angle equals α (dropping terms $\mathcal{O}(\alpha^2)$ and higher). (h) When is this maximum deviation first reached? (i) As a check, verify (for the $\alpha \ll 1$ limit) that Feynman indeed misremembered which way the factor of two had gone in this anecdote about a plate tossed through the air in a Cornell cafeteria: `positron.hep.upenn.edu/p351/feynman`



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XC00*. Optional/extra-credit. If there are extra-credit problems from earlier homework assignments that you didn't have time to do sooner, you can feel free to turn them in with HW13 for full credit. Just clearly indicate for Tanner which problem you're solving.

XC1. Optional/extra-credit. You can do any subset you wish of Taylor's 12.6, 12.7, 12.8, 12.9, 12.10, 12.14, 12.15, 12.32, 12.33, 12.34 (all of which involve some sort of modeling of the DDP or the Logistic Map using Mathematica) and turn them in for extra credit. (Each one counts as an extra-credit problem. Once you've done one of them, it should be easy to do several more.)

XC2. Optional/extra-credit. Remember that if you want to, you can read Chapter 14 (collision theory) for extra credit. If you do so, email me a couple of paragraphs summarizing the key ideas and results of the chapter, to collect your extra credit. You can also do any Chapter 14 problems you like for extra credit.

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