Physics 351 — Monday, February 6, 2017

- Finish handing back HW #2 and quiz #1 — sorry!
- Remember quiz #2 (on hw2) end of class this Wednesday. One sheet of your own hand-written notes OK.
- Remember HW4 due this Friday.
- We are now entering what I think is the most fun part of the course: learning to use the Lagrangian formalism to solve interesting physics problems.
- So let's transition from Calculus of Variations into Lagrangian mechanics.

This proof, from Friday, you'll repeat on HW4. Let's put it to use.

Given: $y(x)$ extremizes $\int \mathrm{d}x\, f(y, y')$

Claim:

$$h \equiv y' \frac{\partial f}{\partial y'} - f(y, y') = \text{constant}$$

Proof:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = y'' \frac{\partial f}{\partial y'} + y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial f}{\partial y'} \frac{\mathrm{d}y'}{\mathrm{d}x}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = y'' \frac{\partial f}{\partial y'} + y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - y' \frac{\partial f}{\partial y} - y'' \frac{\partial f}{\partial y'}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) - y' \frac{\partial f}{\partial y}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = y' \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right) = 0$$

because $y(x)$ must satisfy the Euler-Lagrange equation. QED.

Back to our hanging-chain problem. $f(y, y') = y\sqrt{1 + y'^2}$.

$$h \;\equiv\; y' \frac{\partial f}{\partial y'} - f(y, y') \;=\; \text{constant}$$

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$$h \equiv y'\frac{\partial f}{\partial y'} - f(y, y') = \text{constant}$$

$$f(y,y') = y\sqrt{1+y'^2}$$

$$h = y'\frac{\partial f}{\partial y'} - f = y'\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - y\sqrt{1+y'^2} = h$$

$$yy'^2 - y(1+y'^2) = h\sqrt{1+y'^2}$$

$$yy'^2 - y - yy'^2 = h\sqrt{1+y'^2}$$

$$-y = h\sqrt{1+y'^2} \Rightarrow \left(\frac{y}{h}\right)^2 = 1+y'^2$$

which you could solve e.g. by guessing

$$y = h\cosh\left(\frac{x-B}{h}\right)$$

$$y' = \sinh\left(\frac{x-B}{h}\right) \rightarrow 1+y'^2 = 1+\sinh^2\left(\frac{x-B}{A}\right) = \cosh^2\left(\frac{x-B}{A}\right)$$

$$1+y'^2 = \left(\frac{y}{h}\right)^2 \checkmark$$

If you formally study the Calculus of Variations, the “momentum” trick is called the “first integral” of the E-L equation, and the “energy” trick is called the “Baltrami identity.”

I mentioned them for two reasons: Most importantly, because we will see mechanical analogues of these tricks in the next two weeks. In the first case, “ignorable coordinates” (coordinates not appearing in the Lagrangian) lead to “conserved (generalized) momenta.” In the second case, conservation of energy is expressed (in mechanics) by writing down a Hamiltonian function.

The second reason is that first-order ODEs are generally easier to solve than second-order ODEs, so these two tricks can save effort when using the E-L equation for optimization problems.

If you’re a physicist, you’ll often find that the easiest way to remember a given math result is to remember the analogous physics problem for which it is useful!

Now, a moment we've been waiting 3 weeks for!

In this weekend's reading, you saw that the trajectory of a particle moving in potential $U(x)$ follows the "path of least action," i.e. it follows the path $x(t)$ for which the "action" $S[x(t)]$ is stationary:

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\, \mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\, \mathrm{d}t$$

Feynman points out (XC part of next weekend's reading)

`http://www.feynmanlectures.caltech.edu/II_19.html`

that to be precise, one should really call the Lagrangian approach "the principle of stationary Hamilton's first principal function." But most people say, more concisely, "the principle of least action."

Lagrangian mechanics: Use the Euler-Lagrange equation to find the trajectory $x(t)$ for which the "action" $S[x]$ is stationary.

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\, \mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\, \mathrm{d}t$$

Let's give it a try for a particle of mass m dropped vertically from a short distance x above Earth's surface. For notational simplicity, let the x axis point **vertically upward**. (I should draw this.)

First write down $\mathcal{L}(t, x, \dot{x})$. (Try it!)

Then write the E-L equation: (which variables are which here?)

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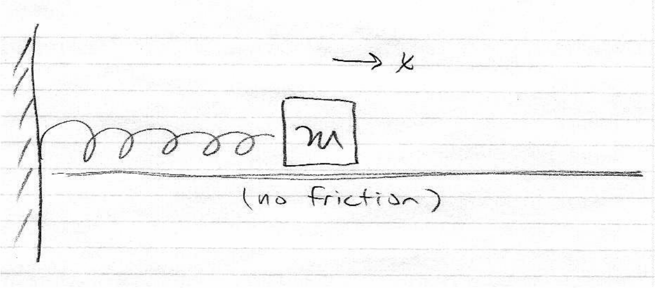
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

Then turn the crank and see what EOM pops out. (Try it!)

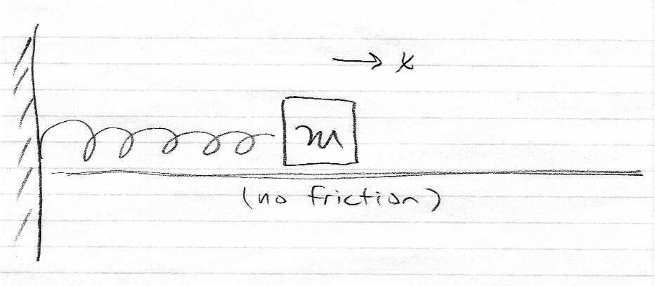


OK, how about a block of mass m moving horizontally on a frictionless table, under the influence of Hooke's-Law potential

$$U = \frac{1}{2}kx^2$$

so $x = 0$ when spring is at its equilibrium length.

Try writing down $\mathcal{L}$, then using $E - L$ equations to find EOM.



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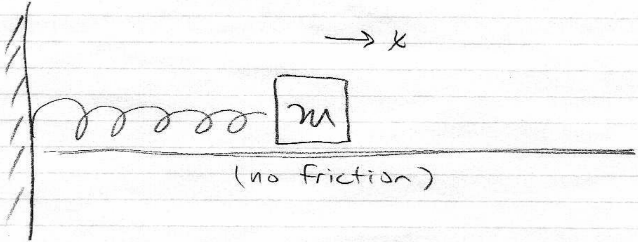
What EOM do we get for a general 1D potential $U(x)$?



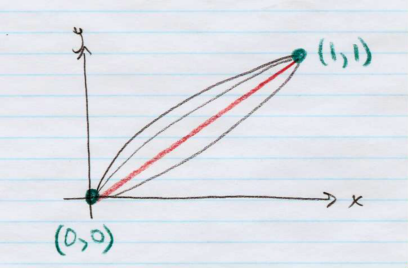
$$T = \frac{1}{2} m \dot{x}^2 \qquad U = \frac{1}{2} k x^2$$

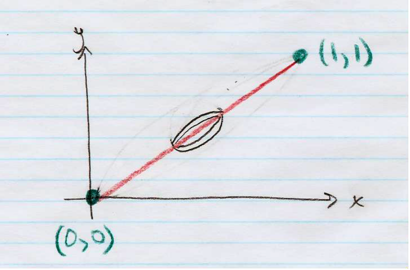
$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \Rightarrow -kx = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x}$$

$$\boxed{m\ddot{x} = -kx}$$ (no surprise!)

When you use the E-L equation to optimize $\int f(x, y, y')\,\mathrm{d}x$, you may be thinking globally, but the E-L equation is acting locally.





The $y(x)$ that makes path length $\int \sqrt{1 + y'^2}\,\mathrm{d}x$ optimal from $(0, 0)$ to $(1, 1)$ will also make the path length optimal from $(0.4, 0.4)$ to $(0.6, 0.6)$ or from $(0.49, 0.49)$ to $(0.51, 0.51)$.

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}$$

is imposing a **local** requirement on $y(x)$. It's acting on each little segment of the curve separately.

Similarly,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

is enforcing $\vec{F} = m\vec{a}$ locally at each step in time. But the effect is that the overall path optimizes the action $S = \int \mathcal{L}\,\mathrm{d}t$.

Lagrangian mechanics: Use the Euler-Lagrange equation to find the trajectory $x(t)$ for which the "action" $S[x]$ is stationary.

$$S[x] \;=\; \int \mathcal{L}(t, x, \dot{x})\,\mathrm{d}t \;=\; \int_{t_i}^{t_f} (T - U)\,\mathrm{d}t$$

As several of you quoted this weekend:

(1) Write down the K.E. and P.E. and hence the Lagrangian $\mathcal{L} = T - U$ using any convenient **inertial** reference frame.

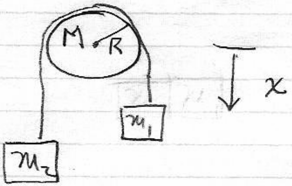
(2) Choose a convenient set of n generalized coordinates q_i and solve for original coords (from step 1) in terms of $q_1 \ldots q_n$.

(3) Rewrite $\mathcal{L}$ in terms of q_i and $\dot{q}_i$.

(4) Write down the n Lagrange equations.

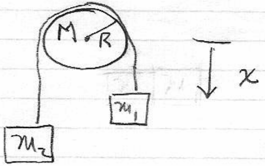
$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Let's work through several examples together, starting from some basic one-variable cases, then becoming more complicated.



Write $L = T - U$ for Atwood machine in which pulley (mass M, radius R) has non-negligible inertia. Take pulley to be a solid cylinder.

After writing down $\mathcal{L}$, use Lagrange eqns to write EOM for $x(t)$, i.e. the expression for $\ddot{x}$.



$$T = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{4}M\dot{x}^2 = \frac{1}{2}\left(m_1+m_2+\frac{M}{2}\right)\dot{x}^2$$

$$U = (m_2-m_1)gx$$

$$L = T-U = \frac{1}{2}\left(m_1+m_2+\frac{M}{2}\right)\dot{x}^2 + (m_1-m_2)gx$$

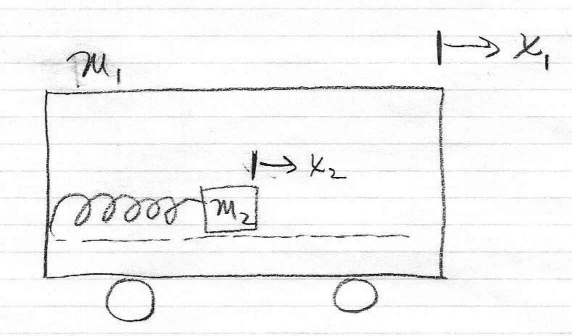
$$\frac{\partial L}{\partial x} = (m_1-m_2)g = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m_1+m_2+\frac{M}{2}\right)\ddot{x}$$

$$\ddot{x} = \frac{(m_1-m_2)}{m_1+m_2+\frac{M}{2}}\,g$$

familiar from HW3

Reading question: "Does the Lagrangian method still work if one chooses generalized coordinates relative to a non-inertial reference frame? If so, is there some precaution one needs to take in writing down the Lagrangian?"

Quoting one of you: "Lagrange's equations are true for any choice of generalized coordinates, even if they are relative to a non-inertial frame. One just has to be careful to write the Lagrangian L=T-U in an inertial frame."



A cart of mass m_1 rolls horizontally without friction. The cart's position is x_1. Inside the cart, a mass m_2 is attached to the wall of the cart with a spring (constant k). The position of m_2 w.r.t. the spring's relaxed position is x_2. So x_2 is w.r.t. the cart, not w.r.t. the ground. Write $\mathcal{L}(t, x_1, \dot{x}_1, x_2, \dot{x}_2)$.

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 - \frac{1}{2} k x_2^2$$

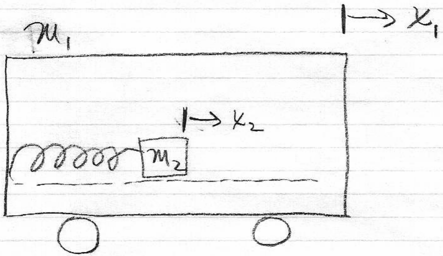
$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = \frac{d}{dt}\left[m_1 \dot{x}_1 + m_2 (\dot{x}_1 + \dot{x}_2) \right]$$

$$0 = (m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = -k x_2 = \frac{d}{dt}\left[m_2 (\dot{x}_1 + \dot{x}_2) \right]$$

$$-k x_2 = m_2 \ddot{x}_1 + m_2 \ddot{x}_2$$

By the way, notice that x_1 is an "ignorable" (a.k.a. "cyclic") coordinate, i.e. $\partial \mathcal{L} / \partial x_1 = 0$. The corresponding conserved quantity is the momentum of the CM, $m_1 \dot{x}_1 + m_2 (\dot{x}_1 + \dot{x}_2)$.



$$x_{cm} = \frac{m_1 x_1 + m_2(x_1 + x_2)}{m_1 + m_2}$$

$$(m_1 + m_2)\, x_{cm} = m_1 x_1 + m_2 (x_1 + x_2)$$

$$= (m_1 + m_2)\, x_1 + m_2 x_2$$

$$(m_1 + m_2)\, \ddot{x}_{cm} = (m_1 + m_2)\, \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

$$\Rightarrow \ddot{x}_{cm} = 0$$

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