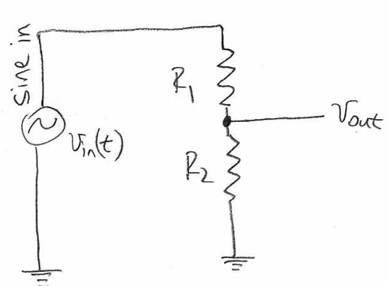
Physics 364, Fall 2014, reading due 2012-09-07.

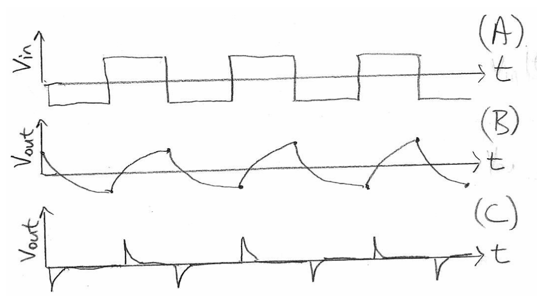
Email your answers to `ashmansk@hep.upenn.edu` by 11pm on Sunday

Course materials and schedule are at `http://positron.hep.upenn.edu/p364`

Assignment: (a) First read carefully through my notes (starting on next page), so that you have a good overview of which points I consider most important for you to absorb from the reading. (b) Then skim through Eggleston's chapter 2 (AC circuits), pausing to read carefully in places where the material is unfamiliar to you. On a first reading, don't get too bogged down in the derivations: if you have time, you can go back and re-read selected details. A few important sections to read carefully are 2.4.1–2.4.4, 2.5.1–2.5.3, 2.6.1–2.6.5.1, and 2.9. (c) Then email me your answers to the questions below.

1. The equation $V_{\rm out} = \frac{R_2}{R_1+R_2} V_{\rm in}$ describes a voltage divider (shown below, left) both for constant $V_{\rm in}$ and for sinusoidal $V_{\rm in}(t)$. **(a)** To turn this circuit into a *high-pass* filter, which resistor (R_1 or R_2) would you replace with a capacitor? **(b)** To make a *low-pass* filter? **(c)** In the low-frequency ($f \to 0$) limit, does a capacitor look like a short-circuit ($Z \to 0$) or an open-circuit ($Z \to \infty$)? **(d)** In the high-frequency ($f \to \infty$) limit? **(e)** How do your answers to parts c and d help you to check your answers for parts a and b? **(f)** If you build a low-pass filter using $R = 1\ {\rm k}\Omega$ and $C = 1\ \mu{\rm F}$, at what frequency f (in Hz, i.e. cycles/second) will the ratio of amplitudes $|V_{\rm out}|/|V_{\rm in}|$ be $\frac{1}{\sqrt{2}} \approx 0.707$? Remember $f = \frac{\omega}{2\pi}$.





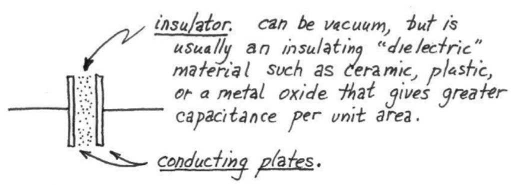
2. In the above-right figure, trace A shows a square-wave input $V_{\rm in}(t)$. **(a)** To turn trace A into $V_{\rm out}(t)$ resembling trace B, which resistor (R_1 or R_2) would you replace with a capacitor? **(b)** To turn trace A into $V_{\rm out}(t)$ resembling trace C? **(c)** If you think "derivative $\sim$ fast change" and "integral $\sim$ slow average," is it the high-pass or the low-pass configuration that is (approximately) integrating $V_{\rm in}$? **(d)** Which configuration (high-pass or low-pass) is (approximately) differentiating $V_{\rm in}$?

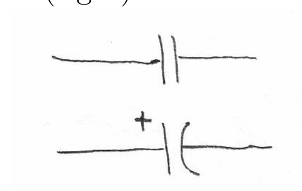
3. Is there anything from this reading assignment that you found confusing and would like me to try to clarify? If you didn't find anything confusing, what topic did you find most interesting?

4. How much time did it take you to complete this assignment?

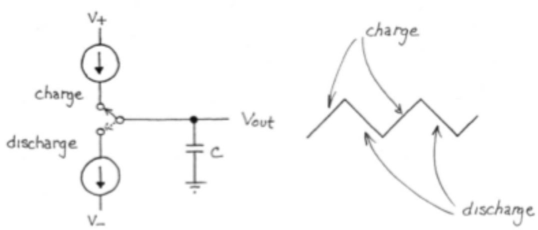
By the end of Lab 3, you will have tried out the oscilloscope and function generator, and you will see your voltage divider respond to a sinusoidal input: the amplitude will reduced by the expected factor $R_2/(R_1 + R_2)$. The key idea so far in the course has been the voltage divider. We used it not only as a circuit fragment performing a desired function (dividing down $V_{\rm in}$ to get $V_{\rm out}$), but also as a means of modeling the output of an imperfect voltage source when a finite load is applied. This week, the key idea will be *impedance*, which generalizes resistance to include capacitors and inductors. Using the impedance concept, we will generalize the voltage divider by replacing one resistor (or sometimes both) with a capacitor (or sometimes a combination of resistors, capacitors, and inductors).

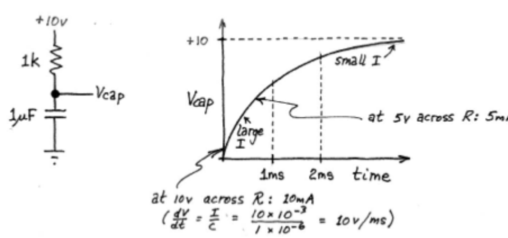
A *capacitor*, shown below (left), is a two-terminal component that stores energy in the electric field between two conducting plates. When potential difference V is applied between the two terminals, the two plates store charges $+Q$ and $-Q$, respectively, where $C = Q/V$ is called the **capacitance** (symbol C, unit F = farad): 1 F = 1 C/V. Applying 1 V across the leads of a 1 μF capacitor stores $\pm 10^{-6}$ coulombs on the two plates. Circuit symbols for capacitors are shown below (right).[1]





Writing $Q = CV$ and differentiating w.r.t. time, we get $I = C\frac{\mathrm{d}V}{\mathrm{d}t}$: the larger the current, the faster V changes. If you could charge and discharge a capacitor with an ideal current source,[2] as shown below (left), you could make a lovely triangle wave. If the current to charge up the capacitor instead comes from the circuit shown below (right),[3] $V_{\rm cap}$ asymptotically approaches the applied voltage: $V_{\rm cap} = (1 - e^{-t/RC})V_{\rm in}$.





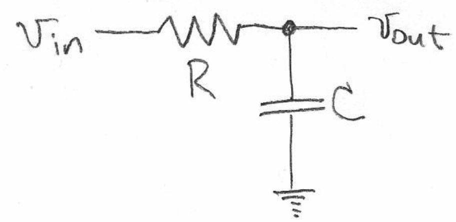
Let's redraw that last circuit so that it looks like our familiar voltage divider, but with the bottom resistor replaced by a capacitor (see below, left). With no load connected at $V_{\rm out}$, the current through the resistor equals the current through the capacitor, and $I = C\frac{\mathrm{d}}{\mathrm{d}t}V_{\rm out}$. Using $IR = (V_{\rm in} - V_{\rm out})$, we get $\frac{\mathrm{d}}{\mathrm{d}t}V_{\rm out} = \frac{1}{RC}(V_{\rm in} - V_{\rm out})$. In the limiting case where $V_{\rm out}$ stays very small ($V_{\rm out} \ll V_{\rm in}$), $V_{\rm out}$ approximates the **integral** of $V_{\rm in}$:

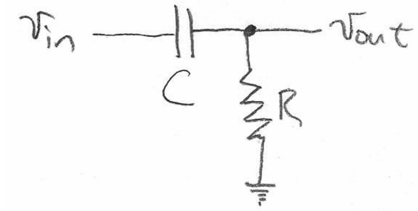
[1] The lower symbol is used only for polarized capacitors, whose dielectric can be damaged if a voltage of the wrong polarity is applied.

[2] Alas, good current sources are far more unusual than good voltage sources.

[3] This figure and several others this week are borrowed from Harvard's course Physics 123.

$V_{\text{out}} \approx \frac{1}{RC} \int V_{\text{in}}\, \mathrm{d}t$ [in $V_{\text{out}} \ll V_{\text{in}}$ limit]. Looking back at the above-right graph, for constant V_{in}, V_{out} starts out looking like the integral of V_{in} (i.e. rising linearly), but the approximation $V_{\text{out}} \ll V_{\text{in}}$ breaks down for $t > 0.1\ RC$ or so.





Now look at the above-right circuit. The current through the capacitor is $I = C \frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{in}} - V_{\text{out}})$, and (with no load connected) $V_{\text{out}} = IR$, so $V_{\text{out}} = RC \frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{in}} - V_{\text{out}})$. In the limiting case $\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} \ll \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{in}}$, the output approximates the **derivative** of the input: $V_{\text{out}} \approx RC\ \mathrm{d}V_{\text{in}}/\mathrm{d}t$ [in $V_{\text{out}} \ll V_{\text{in}}$ limit]. If you look at the bottom V_{out} graph of Eggleston's figure 2.7 and the top V_{out} graph from his figure 2.9, you can see to what degree these two circuits approximate integration and differentiation of V_{in}.

We looked above at RC circuits in the *time domain.* Let's now look in the **frequency domain** at circuits involving resistors, capacitors, and inductors. Ohm's law relates the current through a resistor with the voltage across it: $V = IR$. Now consider placing across a resistor a sinusoidal voltage at frequency $f = \frac{\omega}{2\pi}$: $V(t) = V_p \cos(\omega t)$, where the subscript p means "peak" (i.e. amplitude).

Digression:[4] a circuit's response to sine waves is important because combinations of sines and cosines form solutions to the linear differential equations that describe linear circuits. A linear circuit has the property that its output, when driven by the sum of two input signals, equals the sum of its individual outputs when driven by each input signal in turn. If $O(A)$ represents the output when driven by signal A, then a circuit is linear if $O(A+B) = O(A) + O(B)$. A linear circuit driven by a sine wave at some frequency f always responds with a sine wave at the same frequency f, though in general the phase and amplitude are changed. Circuits designed using ideal resistors, capacitors, and inductors are perfectly linear; even circuits built using real-world resistors, capacitors, and inductors are linear to a remarkable degree. This is why we place so much emphasis on sinusoidal inputs.

OK, back to the main thread: we place across a resistor a sinusoidal voltage at frequency $f = \frac{\omega}{2\pi}$: $V(t) = V_p \cos(\omega t)$. The current through the resistor is $I(t) = V(t)/R = (V_p/R)\cos(\omega t)$. The current and voltage are in phase with one another. So for many expressions of interest, like $V_{\text{out}}(t)/V_{\text{in}}(t)$, the $\cos(\omega t)$ factor just cancels out, which is very convenient.

Now place the same $V(t) = V_p \cos(\omega t)$ across a capacitor: the current through the

[4]Most of this paragraph is taken verbatim from Horowitz & Hill.

capacitor is $I(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\omega V_p \sin(\omega t)$. The current and voltage are 90° out of phase with one another. For sinusoidal $V(t)$, the amplitude of $I(t)$ is still proportional to the amplitude of $V(t)$, but the phase shift makes it impossible for us to cancel out the sines and cosines. That's annoying. It would be great to have a concise way to represent the fact that (for sinusoidal voltage of a given frequency) $I(t)$ is proportional to $V(t)$ but with a phase shift.

Conveniently, the algebra of complex numbers provides a clean notation for representing voltages and currents while keeping track of phase shifts. To represent a sinusoidal voltage with arbitrary phase, we are used to writing $V(t) = V_a\cos(\omega t) + V_b \sin(\omega t)$. Let's instead introduce a complex quantity[5]

$$\mathbf{V}(t) = (V_a - jV_b)e^{j\omega t} = (V_a\cos(\omega t) + V_b\sin(\omega t)) + j(V_a\sin(\omega t) - V_b\cos(\omega t))$$

and we'll agree that any time we want the physical voltage (which must be a real number), we will take the real part of $\mathbf{V}(t)$: $V(t) = \mathcal{R}e(\mathbf{V}(t))$. If I now define a complex amplitude $\mathbf{V}_p = V_a - jV_b$, I can represent $V(t) = A\cos(\omega t)$ by writing $\mathbf{V}_p = A$, and I can represent $V(t) = B\sin(\omega t)$ by writing $\mathbf{V}_p = -jB$. In the first case,

$$V(t) = \mathcal{R}e(\mathbf{V}(t)) = \mathcal{R}e(\mathbf{V}_p e^{j\omega t}) = \mathcal{R}e(A\cos(\omega t) + jA\sin(\omega t)) = A\cos(\omega t),$$

and in the second case,

$$V(t) = \mathcal{R}e(\mathbf{V}(t)) = \mathcal{R}e(\mathbf{V}_p e^{j\omega t}) = \mathcal{R}e(-jB\cos(\omega t) + B\sin(\omega t)) = B\sin(\omega t).$$

Eggleston's text (section 2.6) goes through this formalism in far more detail, in case it is unfamiliar to you.

Returning to the capacitor, we can now write $\mathbf{I}_p = j\omega\mathbf{V}_p$, and by defining a capacitor's **impedance** to be $\mathbf{Z} = -j/(\omega C) = 1/(j\omega C)$, we can write something that looks more like Ohm's law: $\mathbf{V}_p = \mathbf{I}_p\mathbf{Z}$. The fact that $\mathbf{Z}$ for a capacitor is purely imaginary compactly expresses the 90° phase shift between I and V. The impedance for a resistor is still just R.

Now let's look at the same two RC circuits (shown below) in the frequency domain, using our generalized version of Ohm's law. By generalizing resistance to impedance, we can write the response of a **generalized voltage divider** to a sinusoidal input: we find $\mathbf{V}_{\text{out}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1+\mathbf{Z}_2}\mathbf{V}_{\text{in}}$. Replacing R_2 by a capacitor (middle figure), we find

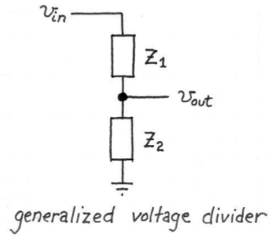
$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\ 2\pi f RC}.$$

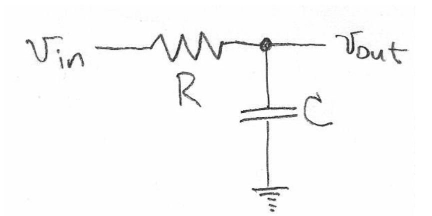
[5]Confusingly, engineers use $j = \sqrt{-1}$ because i is often used to represent small currents. So in electronics, DeMoivre's identity reads $e^{j\theta} = \cos\theta + j\sin\theta$. Note that I will use boldface ($\mathbf{V}$) to denote a complex number, while Eggleston uses a circumflex ($\hat{V}$). Another source of confusion is that engineers use $e^{j\omega t}$ for the time-dependence, while physicists tend to use $e^{-i\omega t}$.

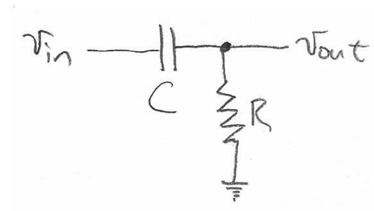
This circuit (middle figure) is called a **low-pass filter**: the lowest-frequency signals pass through unscathed to $V_{\rm out}$, while the highest-frequency signals are shorted to ground by the capacitor. At low frequency ($f \ll \frac{1}{2\pi RC}$), $V_{\rm out} \approx V_{\rm in}$. At high frequency ($f \gg \frac{1}{2\pi RC}$), $V_{\rm out} \ll V_{\rm in}$, and they are 90° apart in phase. The ratio of *amplitudes* is

$$\frac{|\mathbf{V}_{\rm out}|}{|\mathbf{V}_{\rm in}|} = \frac{1}{\sqrt{1+(2\pi f RC)^2}}.$$

At the *corner frequency,* defined as $f_{\rm 3dB} = \frac{1}{2\pi RC}$, the ratio of amplitudes is $\frac{1}{\sqrt{2}} \approx 0.707$. Because $20\log_{10}(1/\sqrt{2}) \approx -3.010$, we say that at $f = f_{\rm 3dB}$, $V_{\rm out}$ is down three decibels (3 dB) from its maximum.







For the right-hand circuit above, where R_1 is replaced by a capacitor, we find

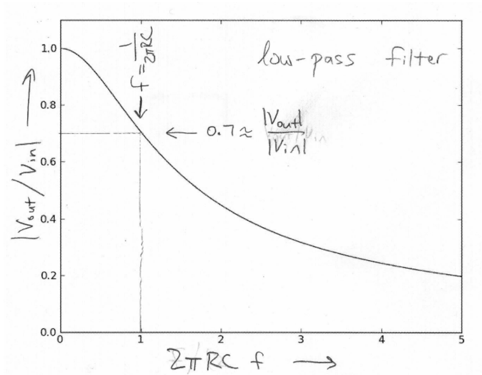
$$\frac{\mathbf{V}_{\rm out}}{\mathbf{V}_{\rm in}} = \frac{R}{1/(j\omega C)+R} = \frac{j\omega RC}{1+j\omega RC} = \frac{2\pi f RC}{2\pi f RC - j},$$

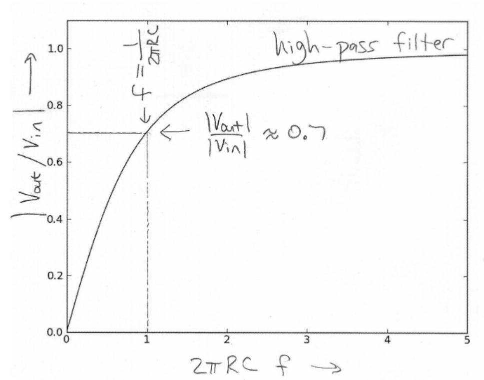
with amplitude ratio

$$\frac{|\mathbf{V}_{\rm out}|}{|\mathbf{V}_{\rm in}|} = \frac{2\pi f RC}{\sqrt{1+(2\pi f RC)^2}}.$$

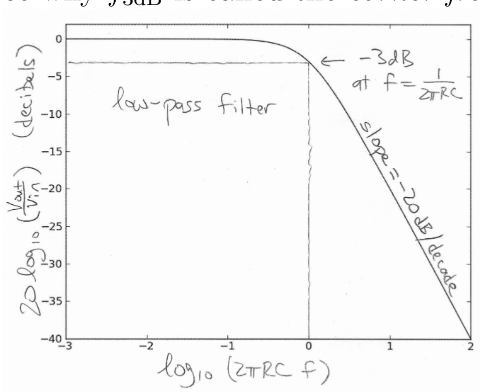
This is a **high-pass filter**: the highest-frequency signals pass through to $V_{\rm out}$, while the lowest-frequency signals are blocked by the capacitor. At high frequency ($f \gg \frac{1}{2\pi RC}$), $V_{\rm out} \approx V_{\rm in}$. At low frequency ($f \ll \frac{1}{2\pi RC}$), $V_{\rm out} \ll V_{\rm in}$, and they are 90° apart in phase. Once again, at $f_{\rm 3dB} = \frac{1}{2\pi RC}$, the ratio of amplitudes is $\frac{1}{\sqrt{2}}$.

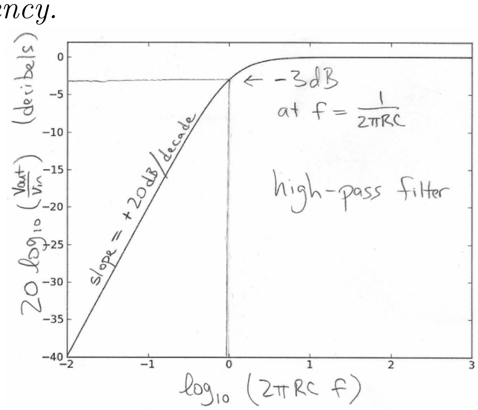
I show below a graph of $|V_{\rm out}|/|V_{\rm in}|$ as a function of $2\pi RC\ f$ (in other words, $f/f_{\rm 3dB}$), for the low-pass filter (left) and the high-pass filter (right).





I show below the same two graphs on a log-log scale (also known as a *Bode plot*). I express $|V_{\text{out}}|/|V_{\text{in}}|$ in decibels by plotting $20\log_{10}(|V_{\text{out}}|/|V_{\text{in}}|)$ on the vertical axis, and I plot $\log_{10}(2\pi RC\ f)$ on the horizontal axis. Notice how the log-log scale clarifies the asymptotic behavior. At high-frequency, the low-pass filter falls off as $1/f$ (a slope of -1 on ordinary log-log axes), and at low-frequency, the high-pass filter rises as f (a slope of $+1$ on ordinary log-log axes). In electronics, one refers to $1/f$ behavior as a slope of -20 dB per decade or alternatively as -6 dB per octave; and one refers to f behavior as a slope of $+20$ dB per decade or as $+6$ dB per octave. This sounds confusing at first, but it makes more sense when you remember that 6 dB means a factor of 2 in amplitude (or a factor of 4 in power), and that 20 dB means a factor of 10 in amplitude (or a factor of 100 in power). An octave is a factor of 2 in frequency, and a decade here means a factor of 10 in frequency. Only on the log-log plot can you see why $f_{3\text{dB}}$ is called the *corner frequency.*





An **inductor** is a two-terminal component that stores energy in a magnetic field. An inductor opposes changes in current by developing a voltage proportional to the rate of change of current: $V(t) = L\frac{\text{d}}{\text{d}t}I(t)$. For sinusoidal signals, $\mathbf{V}_p = j\omega L\ \mathbf{I}_p$. So an inductor L has impedance $\mathbf{Z} = j\omega L$. Replacing R_1 with inductor L in a voltage divider yields $\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}} = R/(R+j\omega L) = 1/(1+j\omega L/R)$, a low-pass filter with $f_{3\text{dB}} = R/(2\pi L)$. Replacing R_2 with inductor L instead yields $\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}} = j\omega L/(R + j\omega L)$, a high-pass filter.

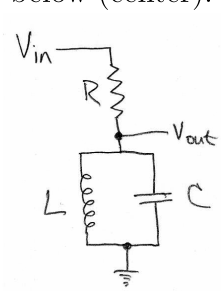
component	impedance	$f \to 0$ limit	$f \to \infty$ limit
resistor	R	R	R
capacitor	$1/(j\omega C)$	open	short
inductor	$j\omega L$	short	open

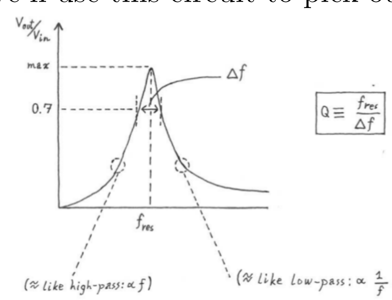
The table above summarizes the complex impedance values used for resistors, capacitors, and inductors. By the way, inductors in series add up just like resistors, and inductors in parallel combine as $\frac{L_1 L_2}{L_1+L_2}$. Capacitors are trickier: capacitances *add* when placed in parallel, and capacitors in series combine as $\frac{C_1 C_2}{C_1+C_2}$. If you remember that *impedances* combine in the same way as ordinary resistances, the upside-down convention for combining capacitors just follows from $Z_C = 1/(j\omega C)$. Never write $C_1 \parallel C_2$, because it is unclear whether you are really talking about capacitors in series or in parallel; instead (if the need ever arises), write $Z_{C1} \parallel Z_{C2}$, which is unambiguous.

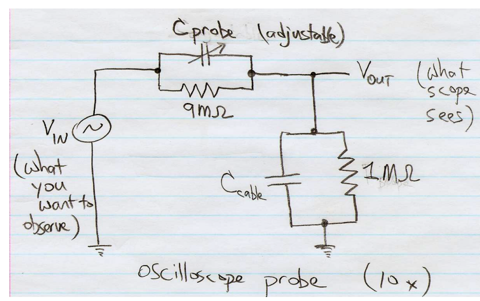
Because I and V are 90° out of phase for a capacitor or an inductor, these devices only store energy; they do not dissipate any power. In the complex notation, the time-averaged power dissipated is $P = \mathcal{R}e(\mathbf{VI}^*) = \mathcal{R}e(\mathbf{IV}^*)$: you multiply voltage by the complex-conjugate of current (or vice-versa) and then take the real part.[6]

Notice the ±90° phase shifts of the low-pass and high-pass filters, for frequencies at which $V_{\rm out} \ll V_{\rm in}$. Recalling that differentiating or integrating a sine shifts its phase by ±90° (since sine and cosine are out of phase by 90°), you can see the "integrator" and "differentiator" we discussed at the beginning, at work here in their domain of validity ($|V_{\rm out}| \ll |V_{\rm in}|$). For the high-pass filter at $f \ll f_{\rm 3dB}$, $V_{\rm out}$ is ahead of $V_{\rm in}$ by 90°; for the low-pass filter at $f \gg f_{\rm 3dB}$, $V_{\rm out}$ lags behind $V_{\rm in}$ by 90°. For the high-pass as $f \to \infty$ or the low-pass at $f \to 0$, the phase shift approaches zero. At $f_{\rm 3dB}$, the phase shift is ±45°.

Suppose we want to pick out one frequency, like that of our favorite radio station, while suppressing other frequencies. If we replace R_2 in our voltage divider with a parallel combination of inductor and capacitor (left figure, below), we get a **bandpass filter**. Using the generalized voltage-divider equation, we get $\mathbf{V}_{\rm out}/\mathbf{V}_{\rm in} = \frac{\mathbf{Z}_2}{R+\mathbf{Z}_2}$, where $\mathbf{Z}_2$ is the impedance of the parallel LC combination: $\mathbf{Z}_2 = j\omega L \parallel \frac{1}{j\omega C}$, which we can rewrite as $\mathbf{Z}_2 = \frac{j\omega L}{1-\omega^2 LC}$. You can see that when $\omega^2 LC = 1$, i.e. when $f = \frac{1}{2\pi\sqrt{LC}}$, $\mathbf{Z}_2$ becomes very large, while for $\omega \to 0$ and for $\omega \to \infty$, $\mathbf{Z}_2 \to 0$. So $V_{\rm out}/V_{\rm in}$ peaks at its **resonant frequency** $f_{\rm res} = \frac{1}{2\pi\sqrt{LC}}$ and has a **bandwidth** $\Delta f \approx \frac{1}{2\pi RC}$, as shown below (center). We'll use this circuit to pick out an AM radio station in a future lab!







You might remember that the input resistance of the oscilloscope is 1 MΩ. This is quite high, but there are times when we might like an even higher $R_{\rm in}$, so that connecting the scope to our circuit alters the observed voltages as little as possible. More importantly, the cables from our circuit to the scope have non-negligible capacitance. For instance, standard RG58 coaxial cable (the kind that normally has BNC connectors on each end) has a capacitance of about 30 pF per foot, or about 100 pF for a 1 m length of cable. The cable capacitance forms a low-pass filter with the scope's input resistance: $f_{\rm 3dB} = \frac{1}{2\pi RC} \approx 1.6$ kHz. While the scope is capable of observing frequencies up to 100 MHz or so, the cable capacitance badly attenuates

[6] Recall that the complex conjugate of $\mathbf{z} = a + jb$ is $\mathbf{z}^* = a - jb$.

and phase-shifts anything above a kHz or so. What to do?! A $\times 10$ **oscilloscope probe** (above-right figure) solves both of these problems at once: it increases R_{in} from 1 MΩ to 10 MΩ and **cancels out** the phase shift and frequency-dependent attenuation. The head of the probe is a 9 MΩ resistor in parallel with an adjustable capacitor. The impedance $\mathbf{Z}_{\text{probe}} = 9\ \text{M}\Omega \parallel \frac{1}{j\omega C_{\text{probe}}}$ forms a voltage divider with the cable+scope impedance $\mathbf{Z}_{\text{c.s.}} = 1\ \text{M}\Omega \parallel \frac{1}{j\omega C_{\text{cable}}}$. By using a tiny screwdriver to adjust C_{probe}, you can arrange that $\mathbf{Z}_{\text{probe}} = 9 \times \mathbf{Z}_{\text{c.s.}}$. (This occurs when $C_{\text{probe}} = \frac{1}{9} C_{\text{cable}}$.) Then if you call the signal you want to observe V_{in}, and you call the signal actually seen by the scope V_{out}, we have $V_{\text{out}}/V_{\text{in}} = \frac{\mathbf{Z}_{\text{c.s.}}}{\mathbf{Z}_{\text{probe}} + \mathbf{Z}_{\text{c.s.}}} = \frac{1}{10}$, with no phase shift or frequency dependence. Also, we have increased R_{in} to 10 MΩ. Jose will show you in class how this adjustment is actually done. It's really sort of a neat trick! You observe a 1 kHz square wave through the probe, and you turn the screw until it really looks like a square wave. (If there is a frequency-dependent attenuation, then the various harmonics will appear in the wrong proportions, and the square wave won't look right. You'll see!)

Finally, two somewhat out-of-place topics that I thought worth mentioning here.[7] First, the voltage across the terminals of a wall socket (in the United States) is 117 volts rms, 60Hz. The amplitude is 165 volts (330 volts pp).

Second: small-signal resistance. We often deal with electronic devices for which I is not proportional to V. In such cases there's not much point in talking about resistance, since the ratio V/I will depend on V, rather than being a nice constant, independent of V. For these devices it is useful to know the slope of the V-vs.-I curve, in other words, the ratio of a small change in applied voltage to the resulting change in current through the device, $\Delta V/\Delta I$ (or $\mathrm{d}V/\mathrm{d}I$). This quantity has the units of resistance (ohms) and substitutes for resistance in many calculations. It is called the small-signal resistance, incremental resistance, or dynamic resistance.[8] This concept is especially useful if you want to superimpose a small AC signal on top of a larger DC voltage. The DC level sets the "operating point" (for example, the average current through a diode), and the slope of the diode's V-vs.-I curve at the operating point determines an effective resistance of the diode for small AC signals that you may superimpose.

[7] I found them both while re-reading Chapter 1 of Horowitz & Hill recently. These two paragraphs are more-or-less taken directly from their text.

[8] The usefulness of small-signal resistance will be clear when we study transistor circuits, a few weeks from now.