1 year $\approx 3.16 \times 10^7$ s

circumference of earth $\approx 40 \times 10^6$ m

speed of light $c = 2.9979 \times 10^8$ m/s

mass of proton or neutron $\approx 1$ amu (“atomic mass unit”) = 1 $\frac{g}{mol} = \frac{0.001 \text{ kg}}{6.022 \times 10^{23}} = 1.66 \times 10^{-27}$ kg

Other unit conversions: try typing e.g. “1 mile in centimeters” or “1 gallon in liters” into google!

average velocity $= \frac{\text{displacement}}{\text{time}}$

average speed $= \frac{\text{distance traveled}}{\text{time}}$

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

g $= 9.8$ m/s$^2$. If x axis points upward, then $a_x = -g$ for free fall near earth’s surface.

If x axis points downward along inclined plane, then $a_x = g \sin \theta$ for object sliding down inclined plane (inclined at angle $\theta$ w.r.t. horizontal).

For constant acceleration:

$v_{x,f} = v_{x,i} + a_x t$

$x_f = x_i + v_{x,i} t + \frac{1}{2} a_x t^2$

$v_{x,f}^2 = v_{x,i}^2 + 2a_x (x_f - x_i)$

Momentum $\vec{p} = m\vec{v}$. Conserved for isolated system: no external pushes or pulls (later we’ll say “forces”). Conservation of momentum in two-body collision implies

$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 v_{2x,f}$

which then implies (for isolated system, two-body collision)

$\frac{\Delta v_{1x}}{\Delta v_{2x}} = -\frac{m_2}{m_1}$
If system is not isolated, then we cannot write $\vec{p}_f - \vec{p}_i = 0$. Instead, we give the momentum imbalance caused by the external influence a name ("impulse") and a label ($\vec{J}$). Then we can write $\vec{p}_f - \vec{p}_i = \vec{J}$.

In chemistry, a calorie is $1 \text{ cal} = 4.18 \text{ J}$. In nutrition, a "food Calorie" is $1 \text{ Cal} = 4180 \text{ J}$.

The energy of motion is kinetic energy:

$$K = \frac{1}{2}mv^2$$

For an elastic collision, kinetic energy $K$ is conserved. For a two-body elastic collision, the relative speed is unchanged by the collision, though obviously the relative velocity changes sign. Thus, for a for a two-body elastic collision along the $x$ axis (Eqn. 5.4),

$$(v_{1x,f} - v_{2x,f}) = -(v_{1x,i} - v_{2x,i})$$

For a totally inelastic collision, the two objects stick together after collision: $\vec{v}_{1f} = \vec{v}_{2f}$. This case is easy to solve, since one variable is eliminated.

In the real world (but not in physics classes), most collisions are inelastic but are not totally inelastic. $K$ is not conserved, but $v_{1z,f} \neq 0$. So you can define a coefficient of restitution, $e$, such that $e = 1$ for elastic collisions, $e = 0$ for totally inelastic collisions, and $0 < e < 1$ for inelastic collisions. Then you can write (though it is seldom useful to do so)

$$(v_{1x,f} - v_{2x,f}) = -e (v_{1x,i} - v_{2x,i})$$

If you write down the momentum-conservation equation for a two-body collision along the $x$ axis,

$$m_1v_{1x,i} + m_2v_{2x,i} = m_1v_{1x,f} + m_2v_{2x,f}$$

and the equation that kinetic energy is also conserved in an elastic collision,

$$\frac{1}{2}m_1v_{1x,i}^2 + \frac{1}{2}m_2v_{2x,i}^2 = \frac{1}{2}m_1v_{1x,f}^2 + \frac{1}{2}m_2v_{2x,f}^2$$

you can (with some effort) solve these two equations in two unknowns. The quadratic equation for energy conservation gives two solutions, which are equivalent to

$$(v_{1x,f} - v_{2x,f}) = \pm (v_{1x,i} - v_{2x,i})$$

In the “+” case, the two objects miss each other, as if they were two trains passing on parallel tracks. The “−” case is the desired solution. In physics, the “other” solution usually means something, even if it is not the solution you were looking for.

Center of mass:
\[
x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}
\]

Center of mass velocity (equals velocity of ZM frame):
\[
v_{ZM,x} = \frac{m_1v_{1x} + m_2v_{2x} + m_3v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}
\]

Convertible kinetic energy: \( K_{conv} = K - \frac{1}{2}mv_{CM}^2 \)

Elastic collision analyzed in ZM (“∗”) frame:
\begin{align*}
  v_{1i,x}^* &= v_{1i,x} - v_{ZM,x}, & v_{2i,x}^* &= v_{2i,x} - v_{ZM,x} \\
  v_{1f,x}^* &= -v_{1i,x}^*, & v_{2f,x}^* &= -v_{2i,x}^* \\
  v_{1f,x} &= v_{1f,x}^* + v_{ZM,x}, & v_{2f,x} &= v_{2f,x}^* + v_{ZM,x}
\end{align*}

Inelastic collision analyzed in ZM frame (restitution coefficient \( e \)):
\begin{align*}
  v_{1f,x}^* &= -ev_{1i,x}^*, & v_{2f,x}^* &= -ev_{2i,x}^*
\end{align*}

Force (newtons: \( 1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2 \)) is rate of change of momentum:
\[
\sum \vec{F} = \frac{d\vec{p}}{dt}
\]

Impulse (change in momentum due to external force):
\[
\vec{J} = \Delta \vec{p} = \int \vec{F}_{\text{ext}} \, dt
\]

Equation of motion for a single object:
\[
\sum \vec{F} = m \, \vec{a}
\]
Equation of motion for CoM of several objects:

$$\sum \vec{F}_{\text{ext}} = m_{\text{total}} \vec{a}_{CM}$$

Gravitational potential energy near earth’s surface ($h =$ height):

$$U_{\text{gravity}} = mgh$$

Force of gravity near earth’s surface (force is $-\frac{dU_{\text{gravity}}}{dx}$):

$$F_x = -mg$$

Potential energy of a spring:

$$U_{\text{spring}} = \frac{1}{2}k(x - x_0)^2$$

Hooke’s Law (force is $-\frac{dU_{\text{spring}}}{dx}$):

$$F_{\text{by spring ON load}} = -k(x - x_0)$$

Work done on a system by an external, nondissipative force in one dimension:

$$W = \int_{x_i}^{x_f} F_x(x)dx$$

which for a constant force reduces to

$$W = F_x \Delta x$$

Power is rate of change of energy (measured in watts: 1 W = 1 J/s):

$$P = \frac{dE}{dt}$$

In one dimension, power delivered by constant external force is

$$P = F_{\text{ext},x}v_x$$
Various ways to write a vector:

\[ \vec{A} = (A_x, A_y) = A_x (1, 0) + A_y (0, 1) = A_x \hat{i} + A_y \hat{j} \]

Can separate into two \( \perp \) vectors that add up to original, e.g.

\[ \vec{A}_x = A_x \hat{i}, \quad \vec{A}_y = A_y \hat{j} \]

\[ \vec{A} = \vec{A}_x + \vec{A}_y \]

Scalar product ("dot product") is a kind of multiplication that accounts for how well the two vectors are aligned with each other:

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = |\vec{A}| \ |\vec{B}| \ \cos(\theta_{AB}) \]

In one dimension, we learned

\[ W = F_x \Delta x \rightarrow \int F_x(x) \ dx \]

Sometimes the force is not parallel to the displacement: for instance, work done by gravity if you slide down a hill. In two dimensions,

\[ W = \vec{F} \cdot \vec{D} = F_x \Delta x + F_y \Delta y \]

which in the limit of many infinitessimal steps becomes

\[ W = \int \vec{F}(\vec{r}) \cdot d\vec{r} = \int (F_x(x, y) \ dx + F_y(x, y) \ dy) \]

Similarly, in two dimensions, power must account for the possibility that force and velocity are not perfectly aligned:

\[ P = \vec{F} \cdot \vec{v} \]

Static friction and kinetic (sometimes called “sliding”) friction:

\[ F^{\text{Static}} \leq \mu_S \ F^{\text{Normal}} \]

\[ F^{\text{Kinetic}} = \mu_K \ F^{\text{Normal}} \]

“normal” & “tangential” components are \( \perp \) to and \( \parallel \) to surface.
For an inclined plane making an angle $\theta$ w.r.t. the horizontal, the normal component of gravity is $F_N = mg \cos \theta$ and the (downhill) tangential component is $mg \sin \theta$. The frictional force on a block sliding down the surface then has magnitude $\mu_K \, mg \cos \theta$ and points uphill if the block is sliding downhill. You have to think about whether things are moving and if so which way they are moving in order to decide which direction friction points and whether the friction is static or kinetic.

For motion in a circle, acceleration has a *centripetel* component that is perpendicular to velocity and points toward the center of rotation. If we put the center of rotation at the origin $(0,0)$ then

$$x = R \cos \theta \quad y = R \sin \theta$$

$$\vec{r} = (R \cos \theta, R \sin \theta) = R (\cos \theta, \sin \theta)$$

The “angular velocity” $\omega$ is the rate of change of the angle $\theta$

$$\omega = \frac{d\theta}{dt}$$

The units for $\omega$ are just $s^{-1}$ (which is the same as radians/second, since radians are dimensionless). Revolutions per second are $\omega/2\pi$, and the period (how long it takes to go around the circle) is $2\pi/\omega$. The velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \omega R (-\sin \theta, \cos \theta), \quad |\vec{v}| = \omega R$$

The magnitude of the centripetal acceleration (the required rate of change of the velocity vector, to keep the object on a circular path) is

$$a_c = \omega^2 R = \frac{v^2}{R}$$

and the centripetal force (directed toward center of rotation) is

$$|\vec{F}_c| = ma_c = m\omega^2 R = \frac{mv^2}{R}$$

Moving in a circle at constant speed (velocity changes but speed does not!) is called *uniform circular motion*. For UCM, $\vec{a} \perp \vec{v}$, and $\omega = \text{constant}$. Then

$$\vec{a} = \frac{d\vec{v}}{dt} = -\omega^2 R (\cos \theta, \sin \theta) = -\frac{v^2}{R} (\cos \theta, \sin \theta)$$

(For non-UCM case where speed is not constant, $\vec{a}$ has an additional component that is parallel to $\vec{v}$.)
We can also consider circular motion with non-constant speed, just as we considered linear motion with non-constant speed. Then we introduce the angular acceleration
\[ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \]
and we can derive results that look familiar but with substitutions
\[ x \to \theta, \quad v \to \omega, \quad a \to \alpha, \quad m \to I, \quad p \to L \]
if \( \alpha \) is constant (which is a common case for constant torque), then:
\[ \theta_f = \theta_i + \omega t + \frac{1}{2} \alpha t^2 \]
\[ \omega_f = \omega_i + \alpha t \]
\[ \omega_f^2 = \omega_i^2 + 2\alpha (\theta_f - \theta_i) \]

Rotational inertia (“moment of inertia”) (see table below):
\[ I = \sum mr^2 \to \int r^2 dm \]

Kinetic energy has both translational and rotational parts:
\[ K = \frac{1}{2} mv_{\text{cm}}^2 + \frac{1}{2} I\omega^2 \]

Angular momentum:
\[ L = I\omega = m v_{\perp} r_{\perp} \]
(where \( \perp \) means the component that does not point toward the “reference” axis — which usually is the rotation axis)

If an object revolves about an axis that does not pass through the object’s center of mass (suppose axis has \( \perp \) distance \( \ell \) from c.o.m.), the rotational inertia is larger, because the object’s c.o.m. revolves around a circle of radius \( \ell \) and in addition the object rotates about its own center of mass. This larger rotational inertia is given by the parallel axis theorem:
\[ I = I_{\text{cm}} + M\ell^2 \]
where \( I_{\text{cm}} \) is the object’s rotational inertia about an axis (which must be parallel to the new axis of rotation) that passes through the object’s c.o.m.

Torque:
\[ \vec{\tau} = \vec{r} \times \vec{F} = rF \sin \theta \]
\[ \tau = I \alpha \]

Work and power:
\[ W = \tau (\theta_f - \theta_i) \]
\[ P = \tau \omega \]

Equilibrium:
\[ \sum \vec{F} = 0, \quad \sum \vec{\tau} = 0 \]

<table>
<thead>
<tr>
<th>configuration</th>
<th>rotational inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>thin cylindrical shell about its axis</td>
<td>( mR^2 )</td>
</tr>
<tr>
<td>thick cylindrical shell about its axis</td>
<td>( \frac{1}{2}(R_i^2 + R_o^2) )</td>
</tr>
<tr>
<td>solid cylinder about its axis</td>
<td>( \frac{1}{2}mR^2 )</td>
</tr>
<tr>
<td>solid cylinder \perp to axis</td>
<td>( \frac{1}{4}mR^2 + \frac{1}{12}m\ell^2 )</td>
</tr>
<tr>
<td>thin rod \perp to axis</td>
<td>( \frac{1}{12}m\ell^2 )</td>
</tr>
<tr>
<td>hollow sphere</td>
<td>( \frac{2}{3}mR^2 )</td>
</tr>
<tr>
<td>solid sphere</td>
<td>( \frac{2}{5}mR^2 )</td>
</tr>
<tr>
<td>rectangular plate</td>
<td>( \frac{1}{12}m(a^2 + b^2) )</td>
</tr>
<tr>
<td>thin hoop about its axis</td>
<td>( mR^2 )</td>
</tr>
<tr>
<td>thin hoop \perp to axis</td>
<td>( \frac{1}{2}mR^2 )</td>
</tr>
</tbody>
</table>
Gravity:

\[ F = \frac{G m_1 m_2}{r^2} \]

where \( \vec{F} \) points along the axis connecting \( m_1 \) to \( m_2 \).

\[ G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \]

is a *universal* constant — the same on Earth, on Mars, in distant galaxies, etc.

\[ g = 9.8 \text{ m/s}^2 = \frac{GM_e}{R_e^2} \]

shows that an apple falling onto Newton’s head results from the same force that governs the motion of the Moon around Earth, Earth around the Sun, etc.

For an orbit, gravity provides the centripetal force, so

\[ \frac{m v^2}{R} = \frac{G M m}{R^2} \]

Gravitational potential energy for objects 1 and 2 is

\[ U = -\frac{G m_1 m_2}{r} \quad \text{(note the sign)} \]

which \( \to 0 \) as \( r \to \infty \). The objects are *bound* if \( K + U < 0 \).

If \( K + U \geq 0 \), they escape each other. They just barely escape if \( K + U = 0 \)

\[ \frac{1}{2} m v_{\text{escape}}^2 = \frac{GM}{R} \]

in which case \( K \to 0 \) when \( R \to \infty \).

G.P.E. of e.g. a spacecraft of mass \( m \) in the field of two large objects (e.g. earth and moon) of mass \( M_1 \) and \( M_2 \):

\[ U = -\left( \frac{G M_1 m}{R_{M1,m}} + \frac{G M_2 m}{R_{M2,m}} \right) \]

(needed for Problems 12 and (if you want to be precise) 10)

For a central force that goes like \( F \propto 1/R^2 \), the forces from a uniform spherical shell add (if you’re outside the shell) up to one force directed from the center of the shell. So a rigid sphere attracts you as if it were a point mass.
If you’re inside the shell, the sum of the forces adds up to zero.

Static equilibrium (all forces/torques acting ON the object sum to zero):

\[ \sum F_x = 0, \quad \sum F_y = 0, \quad \sum \tau = 0 \]

Young’s modulus: \( \frac{\Delta L}{L_0} = \frac{1}{E} \left( \frac{\text{force}}{\text{area}} \right) \)

**Oscillations** (mostly illustrate using mass and spring). Combining \( F = ma \) with \( F = -kx \), we get \( m\ddot{x} = -\omega^2 x \), which has solution

\[ x = A \cos(\omega t + \phi), \quad v_x = \dot{x} = -\omega A \sin(\omega t + \phi) \]

You can also write it in terms of frequency \( f \), using \( \omega = 2\pi f \):

\[ x = A \cos(2\pi f t + \phi), \quad v_x = -2\pi f A \sin(2\pi ft + \phi) \]

where \( f \) is frequency (cycles per second) and \( \omega \) is “angular frequency” (radians per second). When a frequency is given in Hz (hertz), it always means \( f \), not \( \omega \). The A above middle C on a piano has frequency \( f = 440 \) Hz, and the buzzing you hear from electrical appliances is 60 Hz (or a small-integer multiple, e.g. 120 Hz).

So frequency is \( f = \frac{\omega}{2\pi} \) (how many times the thing vibrates per second), period is \( T = \frac{1}{f} = \frac{2\pi}{\omega} \) (how many seconds elapse per vibration). The maximum displacement is *amplitude* \( A \), measured in meters. The maximum speed is \( \omega A \) (units are meters/second). The initial phase, \( \phi \), tells you where you are in the oscillation at \( t = 0 \). If at \( t = 0 \) you have \( x > 0 \) but \( v_x = 0 \), then \( \phi = 0 \). If at \( t = 0 \) you have \( v_x > 0 \) but \( x = 0 \), then \( \phi = \pi/2 \) (90°). The energy is \( K + U = \frac{1}{2} m\omega^2 A^2 \).

For a pendulum, you get \( \theta = A \cos(\omega t + \phi) \), with \( \omega = \sqrt{g/\ell} \). This requires two approximations: first, that \( \theta \) is small enough that \( \sin \theta \approx \theta \); second, that the mass is concentrated at a point at the end of the string, i.e. that the shape of the mass does not contribute to the rotational inertia of the pendulum.

If the second approximation does not hold (e.g. the rod is about as heavy as the mass on the end), then you have a “physical pendulum” with \( \omega = \sqrt{mg\ell_{cm}/I} \), where \( \ell_{cm} \) is the distance from the pivot to the CoM, and \( I \) is the rotational inertia about the **pivot** (not about the CoM).

For damped oscillations, the energy decays away with a factor \( e^{-t/\tau} \), where the symbol \( \tau \) in this case means “decay time constant,” (not torque!). The quality factor \( Q = 2\pi f \tau \) tells you how many oscillation periods it takes for the oscillator to lose a substantial fraction \( 1 - e^{-2\pi} \approx 99.8\% \) of its stored energy.
For two springs connected in parallel (side-by-side), $k = k_1 + k_2$. For two springs connected in series (end-to-end), $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$, or equivalently $k = \frac{k_1 k_2}{k_1 + k_2}$. 